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INTRINSIC ULTRACONTRACTIVITY OF THE FEYNMAN-KAC SEMIGROUP FOR RELATIVISTIC STABLE PROCESSES

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Abstract. Let X_t be the relativistic α-stable process in $\mathbf{R}^d$, $\alpha \in (0,2)$, $d > \alpha$, with infinitesimal generator $H_0^{(\alpha)} = -((-\Delta + m^{2/\alpha})^{\alpha/2} - m)$. We study intrinsic ultracontractivity (IU) for the Feynman-Kac semigroup T_t for this process with generator $H_0^{(\alpha)} - V$, $V \geq 0$, V locally bounded. We prove that if $\lim_{|x|\to\infty} V(x) = \infty$, then for every $t > 0$ the operator T_t is compact. We consider the class $\mathcal{V}$ of potentials V such that $V \geq 0$, $\lim_{|x|\to\infty} V(x) = \infty$ and V is comparable to the function which is radial, radially nondecreasing and comparable on unit balls. For V in the class $\mathcal{V}$ we show that the semigroup T_t is IU if and only if $\lim_{|x|\to\infty} V(x)/|x| = \infty$. If this condition is satisfied we also obtain sharp estimates of the first eigenfunction ϕ_1 for T_t. In particular, when $V(x) = |x|^\beta$, $\beta > 0$, then the semigroup T_t is IU if and only if $\beta > 1$. For $\beta > 1$ the first eigenfunction $\phi_1(x)$ is comparable to

$$\exp(-m^{1/\alpha}|x|)\,(|x|+1)^{(-d-\alpha-2\beta-1)/2}.$$

1. Introduction

The purpose of this paper is to study the Feynman-Kac semigroup for the relativistic α-stable process X_t on $\mathbf{R}^d$, $\alpha \in (0,2)$. This process is a Markov process with independent and homogeneous increments and characteristic function of the form

$$\mathbf{E}^0(\exp(i\xi X_t)) = \exp\left(-t((m^{2/\alpha} + |\xi|^2)^{\alpha/2} - m)\right),$$

where $\xi \in \mathbf{R}^d$, $m > 0$, $t > 0$. In the entire paper we assume that $d > \alpha$. As usual $\mathbf{E}^x$, $x \in \mathbf{R}^d$, denotes the expected value for the process starting from $x \in \mathbf{R}^d$.

The Feynman-Kac semigroup T_t, $t > 0$, for X_t and measurable, locally bounded potential $0 \leq V(x) < \infty$ is defined as follows:

$$T_t(f)(x) = \mathbf{E}^x\left(\exp\left(-\int_0^t V(X_s)ds\right) f(X_t)\right),\ x \in \mathbf{R}^d,\ f \in L^2(\mathbf{R}^d). \tag{1.1}$$

The generator of this semigroup is the Schrödinger operator $H^{(\alpha)} = H_0^{(\alpha)} - V$, where

$$H_0^{(\alpha)} = -((-\Delta + m^{2/\alpha})^{\alpha/2} - m).$$

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In this paper we study the Feynman-Kac semigroup for the generator $H^{(\alpha)} = H_0^{(\alpha)} - V$ by using methods of stochastic processes. Although proofs are rather complicated they are quite general and can be applied to many other operators e.g. $-(-\Delta)^{\alpha/2}$, which are generators of symmetric α-stable processes. It is worth pointing out that Feynman-Kac semigroups for Markov processes (especially for symmetric α-stable processes) have been widely studied ([Z], [BB1], [BB2], [CS1], [CS2]).

The relativistic α-stable process has been introduced and studied in [R]. For $\alpha = 1$ this process has been studied in [CMS] (see also [C1], [Ba1] and [Ba2]). For $\alpha = 1$ the generator of this process has the form

$$H_0^{(1)} = -(\sqrt{-\Delta + m^2} - m)$$

and $-H_0^{(1)}$ is called relativistic Hamiltonian. As explained in [CMS] this operator corresponds to the kinetic energy of a relativistic particle with mass m. If p is the momentum of the particle, then its relativistic kinetic energy is given by $E = \sqrt{p^2 + m^2}$. In the process of quantization the momentum p is replaced by the differential operator $-i\nabla$, and the quantum analog of the relativistic kinetic energy is the free relativistic Hamiltonian $-H_0^{(1)}$.

There are many problems in quantum mechanics which can be formulated in terms of such generators. For example they were investigated by E. Lieb in difficult problems concerning the stability of relativistic matter. There exists an important literature on properties of relativistic Hamiltonians ([L], [H], [DL], [Db], [F], [FL], [LY]).

Now we come back to formulating results for the Feynman-Kac semigroup T_t of the relativistic α-stable process. Let us recall that we assume in this paper that the potential V which appears in the definition of the Feynman-Kac semigroup satisfies $0 \le V(x) < \infty$. The Feynman-Kac semigroup T_t is given by the kernel $u(t, x, y)$, that is,

$$T_t f(x) = \int_{\mathbf{R}^d} u(t, x, y) f(y) dy, \ x \in \mathbf{R}^d, \ f \in L^2(\mathbf{R}^d).$$

For each $t > 0$ the kernel $u(t, x, y)$ is continuous and bounded on $\mathbf{R}^d \times \mathbf{R}^d$. For any $t > 0$, x, $y \in \mathbf{R}^d$ the kernel is strictly positive. The proof of these properties is standard. It is similar to proofs for the classical Feynman-Kac semigroup (see e.g. [CZ]). For the convienience of the reader we write the short proof of properties of $u(t, x, y)$ in Lemma 3.1.

Our first result gives an easy criterion for compactness of operators T_t.

Theorem 1.1. *If $V(x) \longrightarrow \infty$ as $|x| \longrightarrow \infty$, then for all $t > 0$ the operators T_t are compact. If there exists a set consisting of an infinite number of disjoint unit balls such that $V(x)$ is bounded on this set, then for all $t > 0$ the operators T_t are not compact.*

From now on we will assume that $V(x) \longrightarrow \infty$ as $|x| \longrightarrow \infty$. The properties of $u(t, x, y)$ and general theory of semigroups for compact operators gives the following standard results. There exists an orthonormal basis in $L^2(\mathbf{R}^d)$ of eigenfunctions $\{\phi_n\}_{n=1}^\infty$ with corresponding eigenvalues $\{e^{-\lambda_n t}\}_{n=1}^\infty$ satisfying $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ and $\lim_{n \to \infty} \lambda_n = \infty$. That is, $T_t \phi_n = e^{-\lambda_n t} \phi_n$. All ϕ_n are continuous and bounded. The first eigenfunction ϕ_1 is strictly positive.

The most important result of this paper concerns *intrinsic ultracontractivity* (IU) for the semigroup T_t. IU was introduced by E. B. Davies and B. Simon in [DS]. The semigroup T_t is called intrinsically ultracontractive if and only if for any $t > 0$ there exists a constant C_t such that for all $x,y \in \mathbf{R}^d$

$$u(t,x,y) \le C_t \phi_1(x)\phi_1(y). \tag{1.2}$$

This definition comes from [DS], Theorem 3.2(iv), which presents many equivalent conditions for IU. It is well known that the upper bound inequality implies the lower bound inequality (see [DS], Theorem 3.2, proof of (iv) $\Rightarrow$ (v)). Therefore IU may also be formulated in the following way. The semigroup T_t is called intrinsically ultracontractive if and only if for any $t > 0$ there exist constants $C_t, c_t > 0$ such that for all $x,y \in \mathbf{R}^d$

$$c_t \phi_1(x)\phi_1(y) \le u(t,x,y) \le C_t \phi_1(x)\phi_1(y). \tag{1.3}$$

There are many other equivalent conditions for IU (see [DS], [B]).

Let us point out that in [DS] it is assumed that $\int_{\mathbf{R}^d} u(t,x,x)\,dx < \infty$, and we do not assume this apriori. Nevertheless we do not use this assumption anywhere, and also the proof of (1.2) $\Rightarrow$ (1.3) does not use this assumption.

In this paper we will check IU using the following conditions which may be studied using probabilistic methods.

Condition 1.2. *There exists an open, bounded and nonempty set D such that for any $t > 0$ there is a constant $c_{t,D} > 0$ such that for any $x \in \mathbf{R}^d$,*

$$T_t(1_{\mathbf{R}^d})(x) \le c_{t,D} T_t(1_D)(x).$$

Condition 1.3. *For any open, bounded and nonempty set D and for any $t > 0$ there is a constant $c_{t,D} > 0$ such that for any $x \in \mathbf{R}^d$,*

$$T_t(1_{B(x,1)})(x) \le c_{t,D} T_t(1_D)(x).$$

Condition 1.2 implies IU. We will show this at the end of Section 3. The fact that Condition 1.2 implies IU is rather well known (see e.g. [BD], Lemma 1.4). However we could not find in the literature the direct proof for the Feynman-Kac semigroup. Therefore we decided to provide the brief proof.

Also at the end of Section 3 we will show that IU implies Condition 1.3. This condition will be used to show that for some potential V the semigroup is not IU.

IU has been introduced in [DS] for very general semigroups. Important examples of such semigroups are the semigroups of elliptic operators H_0 and the semigroups for Schrödinger operators $H = H_0 - V$ both on $\mathbf{R}^d$, as well as on domains D (with Dirichlet boundary conditions). IU for such semigroups has been widely studied (see e.g. [B], [Da] [D], [BD]). IU has also been studied for semigroups generated by $-(-\Delta)^{\alpha/2}$ (see e.g. [K], [CS1], [CS2]).

The classical result for Feynman-Kac semigroups T_t on $\mathbf{R}^d$ generated by $H = \Delta - V$ is the following fact (Corollary 4.5.5, Theorem 4.5.11 and Corollary 4.5.8 in [D]). If $V(x) = |x|^\beta$ (dimension $d \ge 1$), then T_t is IU iff $\beta > 2$. Moreover for $\beta > 2$ we have

$$cf(x) \le \phi_1(x) \le Cf(x), \ |x| > 1,$$

where c and C are positive constants, and

$$f(x) = |x|^{-\beta/4+(d-1)/2} \exp\left(-\frac{2}{2+\beta}|x|^{1+\beta/2}\right).$$

There are of course many other results of similar type (see e.g. Theorems 6.1, 6.3, 6.4 in [DS]).

Now we will define the class $\mathcal{V}$ of potentials which we will investigate in this paper. First we need the definition of the auxiliary class of functions $\mathcal{L}$.

Definition 1.4. We say that a function $L : [0, \infty) \longrightarrow [0, \infty)$ belongs to the class $\mathcal{L}$ if

(1) L is nondecreasing,
(2) $\lim_{t \to \infty} L(t) = \infty$,
(3) there exists $\tilde{c} \geq 1$ such that for all $t \geq 0$

$$L(t+1) \leq \tilde{c} L(t) + \tilde{c}.$$

Definition 1.5. We say that the potential $V : \mathbf{R}^d \longrightarrow [0, \infty)$ belongs to the class $\mathcal{V}$ if there exists a function $L \in \mathcal{L}$ and a constant $C > 0$ such that for any $x \in \mathbf{R}^d$ we have

$$L(|x|) \leq V(x) \leq C L(|x|) + C.$$

Roughly speaking $V \in \mathcal{V}$ if $V \geq 0$, $\lim_{|x| \to \infty} V(x) = \infty$ and V is comparable to the function which is radial, radially nondecreasing and comparable on unit balls. Typical examples of functions belonging to $\mathcal{V}$ are $V(x) = |x|^\beta$ for $\beta > 0$, $V(x) = |x|^\beta \ln^\gamma(|x| + 2)$ for $\beta \geq 0$ and $\gamma > 0$, and $V(x) = e^{\beta |x|}$ for $\beta > 0$. On the other hand $V(x) = \exp(|x|^\beta)$, $\beta > 1$, does not belong to $\mathcal{V}$ because $\exp(|x|^\beta)$ and $\exp((|x| + 1)^\beta)$ are not comparable.

The main result of this paper is the following theorem.

Theorem 1.6. *Assume that the potential V belongs to class $\mathcal{V}$. Then the Feynman-Kac semigroup T_t with such potential (defined by* (1.1)*) is intrinsically ultracontractive if and only if*

$$\lim_{|x| \to \infty} \frac{V(x)}{|x|} = \infty. \tag{1.4}$$

Moreover, if (1.4) *holds, then there exist $c_1 = c_1(d, \alpha, m, V) > 0$ and $c_2 = c_2(d, \alpha, m, V) > 0$ such that for any $x \in \mathbf{R}^d$ we have*

$$\frac{c_1 \exp(-m^{1/\alpha}|x|)}{(|x| + 1)^{(d+\alpha+1)/2}(V(x) + 1)} \leq \phi_1(x) \leq \frac{c_2 \exp(-m^{1/\alpha}|x|)}{(|x| + 1)^{(d+\alpha+1)/2}(V(x) + 1)}. \tag{1.5}$$

In particular for potentials $V(x) = |x|^\beta$, the semigroup T_t is IU if and only if $\beta > 1$. When $\beta > 1$ there exists $c_1 = c_1(\alpha, d, m, \beta) > 0$ and $c_2 = c_2(\alpha, d, m, \beta) > 0$ such that

$$\frac{c_1 \exp(-m^{1/\alpha}|x|)}{(|x| + 1)^{(d+\alpha+2\beta+1)/2}} \leq \phi_1(x) \leq \frac{c_2 \exp(-m^{1/\alpha}|x|)}{(|x| + 1)^{(d+\alpha+2\beta+1)/2}}.$$

This gives control on the growth of $u(t, x, y)$ (cf. (1.3)).

The rest of the paper is organized as follows. In Section 2, Preliminaries, we set notation and present various facts which are needed in the sequel. In Section 3 we prove Theorem 1.1 which gives criterion for compactness of T_t. In Section 4 we prove estimates of transition density for the killed process. These estimates are needed to prove the main result. Nevertheless it seems that these estimates were not known before and are interesting in themselves. In Section 5 we prove the main result of the paper, Theorem 1.6. Section 5 is the most important and difficult part of this paper. We use probabilistic methods to prove intrinsic ultracontractivity.

One of the key steps in the proof of the main theorem is Lemma 5.9. The main idea of the proof of this lemma is taken from [BK], Lemma 4.5.

2. Preliminaries

Let $\mathbb{N} = \{1, 2, \dots\}$ denote the set of natural numbers. Let $d \geq 1$. By $|x|$ we will denote the Euclidean norm of in $\mathbf{R}^d$, and by $|A|$ the d-dimensional Lebesgue measure of set A. For any subset $U \in \mathbf{R}^d$ we will denote its complement by U^c. Furthermore for $x \in \mathbf{R}^d$, $r > 0$, we put $B(x, r) = \{y \in \mathbf{R}^d : |x - y| < r\}$. For any A,$B \subset \mathbf{R}^d$, $t > 0$, we denote $\mathrm{dist}(A, B) = \inf\{x \in A, y \in B : |x - y|\}$, $tA = \{tx : x \in A\}$, $\delta_A(x) = \mathrm{dist}(x, \partial A)$.

We will write $c = c(\alpha, \beta, \dots, \gamma)$ to indicate the dependence of a constant c on parameters, functions, etc. All constants in this paper depend on the process, and (if applicable) on the potential, thus we will omit dependence on α, d, m, and V. The constants may change their value from one use to the next, even on the same line in the same formula. However, the set of parameters on which a constant may depend will not change from one use to another. The constants will always be assumed finite and strictly positive.

From now on let $\alpha \in (0, 2)$. We will follow terminology and notation from [R] most of the time.

The density of the transition probability for X_t is given by the formula

$$\mathbf{P}^x(X_t \in A) = \int_A p(t, x, y)dy.$$

It is well known (Lemma 3 from [R]) that for all $t > 0$ the density $p(t, x, y)$ is bounded. The density of the Levy measure for the relativistic α-stable process, called $\nu(x)$, is equal to (Lemma 2 from [R])

$$\nu(x) = \frac{c}{|x|^{d+\alpha}} e^{-m^{1/\alpha}|x|} \varphi(m^{1/\alpha}|x|), \tag{2.1}$$

for $x \in \mathbf{R}^d$, $|x| > 0$, where

$$\varphi(\xi) = \int_0^\infty e^{-v} v^p (\xi + v/2)^p dv, \;\; \xi \geq 0, \; p = \frac{d + \alpha - 1}{2}, \tag{2.2}$$

and $c = \Gamma((d + \alpha)/2)/(\pi^{d/2} 2^{-\alpha/2} |\Gamma(-\alpha/2)| \varphi(0))$.

By τ_D we will denote the first exit time from the open set D, i.e., $\tau_D = \inf\{t > 0 : X_t \notin D\}$. The exit time has the following property: $\mathbf{P}^x(\tau_D = t) = 0$ for all $t > 0$.

By $p_D(t, x, y)$ we denote the density of the process killed on exiting the set D:

$$p_D(t, x, y) = p(t, x, y) - \mathbf{E}^x(\tau_D \leq t; p(t - \tau_D, X(\tau_D), y)), \tag{2.3}$$

for x,$y \in D$, and $p_D(t, x, y) = 0$ everywhere else. For the open bounded set D we will denote by $G_D(x, y)$ the Green function for the set D equal to $G_D(x, y) = \int_0^\infty p_D(t, x, y)dt$.

For an open set $D \subset \mathbf{R}^d$ and $x \in \mathbf{R}^d$, the distribution $\mathbf{P}^x(\tau_D < \infty, X(\tau_D) \in \cdot)$ will be called the relativistic α-harmonic measure for D. The following Ikeda-Watanabe formula recovers the relativistic α-harmonic measure for the set D from the Green function.

Proposition 2.1 ([IW]). *Assume that D is an open, nonempty, bounded subset of $\mathbf{R}^d$, and A is a Borel set such that* $\operatorname{dist}(D,A)>0$. *Then*

$$\mathbf{P}^x(X(\tau_D)\in A, \tau_D<\infty)=\int_D G_D(x,y)\int_A \nu(y-z)dzdy,\ x\in D. \tag{2.4}$$

Now we prove some estimates for $p(t,x,y)$ and $\nu(x)$, which are crucial in further considerations.

Lemma 2.2. *For any $x,y\in\mathbf{R}^d$ and $t>0$ there exist constants $c_1>0$ and $c_2>0$, such that*

$$p(t,x,y)\le c_1e^{mt}\min\left\{\frac{t}{|x-y|^{d+\alpha}}e^{-c_2|x-y|},t^{-d/\alpha}\right\}.$$

Proof. Let us recall our convention that all constants (in particular c_1, c_2) may depend on α, d, m, V, and we omit this dependence in notation. Moreover, all constants are strictly positive. The following inequality is the consequence of formula (8) in the proof of Lemma 2 from [R]:

$$p(t,x,y)\le cte^{mt}\int_0^\infty \frac{1}{u^{\frac{d+\alpha+2}{2}}}e^{-m^{2/\alpha}u}e^{\frac{-|x-y|^2}{4u}}du. \tag{2.5}$$

We have

$$\int_0^{|x-y|}\frac{1}{u^{\frac{d+\alpha+2}{2}}}e^{-m^{2/\alpha}u}e^{\frac{-|x-y|^2}{4u}}du\le e^{-|x-y|/8}\int_0^{|x-y|}\frac{1}{u^{\frac{d+\alpha+2}{2}}}e^{\frac{-|x-y|^2}{8u}}du,$$
$$\int_{|x-y|}^\infty\frac{1}{u^{\frac{d+\alpha+2}{2}}}e^{-m^{2/\alpha}u}e^{\frac{-|x-y|^2}{4u}}du\le e^{-m^{2/\alpha}|x-y|}\int_{|x-y|}^\infty\frac{1}{u^{\frac{d+\alpha+2}{2}}}e^{\frac{-|x-y|^2}{8u}}du.$$

Thus

$$p(t,x,y)\le cte^{mt}e^{-c_2|x-y|}\int_0^\infty\frac{1}{u^{\frac{d+\alpha+2}{2}}}e^{\frac{-|x-y|^2}{8u}}du.$$

Substituting $v=|x-y|^2/(8u)$, the last expression equals

$$cte^{mt}e^{-c_2|x-y|}\int_0^\infty\frac{v^{\frac{d+\alpha-2}{2}}}{|x-y|^{d+\alpha}}e^{-v}dv=\frac{c_1te^{mt}}{|x-y|^{d+\alpha}}e^{-c_2|x-y|}.$$

The second estimate (by $t^{-d/\alpha}$) is a consequence of Lemma 5 from [R] and well-known estimates for transition density for classical α-stable processes. □

We obtained the following explicit formula for the asymptotic behavior of $\nu(x)$.

Lemma 2.3. *Let $\varepsilon>0$. There exist constants c_ε and C_ε, such that for $|x|>\varepsilon$ we have*

$$\frac{c_\varepsilon}{|x|^{\frac{d+\alpha+1}{2}}}e^{-m^{1/\alpha}|x|}\le\nu(x)\le\frac{C_\varepsilon}{|x|^{\frac{d+\alpha+1}{2}}}e^{-m^{1/\alpha}|x|}. \tag{2.6}$$

Proof. To get the asymptotic behavior of $\nu(x)$ we need to estimate φ from (2.2). Assume $\xi\ge\varepsilon m^{1/\alpha}$. We divide φ into two parts,

$$I_1=\int_0^\xi e^{-v}v^p(\xi+v/2)^pdv,$$
$$I_2=\int_\xi^\infty e^{-v}v^p(\xi+v/2)^pdv.$$

We have

$$I_2 \le \int_\xi^\infty e^{-v} v^p (2v)^p dv \le 2^p \int_0^\infty e^{-v} v^{2p} dv,$$

so I_2 is bounded. On the other hand,

$$I_1 \le (2\xi)^p \int_0^\xi e^{-v} v^p dv \le (2\xi)^p \int_0^\infty e^{-v} v^p dv = c|\xi|^p,$$

$$I_1 \ge |\xi|^p \int_0^\xi e^{-v} v^p dv \ge |\xi|^p \int_0^{\varepsilon m^{1/\alpha}} e^{-v} v^p dv = c|\xi|^p. \qquad \square$$

Lemma 2.4. *For any $x \in \mathbf{R}^d$ we have*

$$\nu(x) \le \frac{c}{|x|^{d+\alpha}}.$$

Proof. The lemma is a consequence of formulas (2.1) and (2.2). $\square$

Now we prove generalizations of the Ikeda-Watanabe formula (Proposition 2.1).

Proposition 2.5. *Assume that D is an open, nonempty, bounded subset of $\mathbf{R}^d$, and A is a Borel set such that $A \subset D^c \setminus \partial D$. Then*

$$\begin{aligned} &\mathbf{P}^x(X(\tau_D) \in A, t_1 < \tau_D < t_2) \\ &\qquad = \int_D \int_{t_1}^{t_2} p_D(s,x,y) ds \int_A \nu(y-z) dz dy, \end{aligned} \tag{2.7}$$

where $x \in D$, $0 \le t_1 < t_2 \le \infty$.

Proof. It is sufficient to consider only the case when $t_2 = \infty$. At first assume that $\operatorname{dist}(A, D) > 0$. Using strong Markov property we have, for any $t > 0$ and $x \in D$,

$$\begin{aligned} &\mathbf{P}^x(X(\tau_D) \in A, t < \tau_D < \infty) \\ &\quad = \mathbf{E}^x \left(t < \tau_D; \mathbf{P}^{X_t}(X(\tau_D) \in A, \tau_D < \infty)\right) \\ &\quad = \mathbf{E}^x \left(t < \tau_D; \int_D G_D(X_t, y) \int_A \nu(y-z) dz dy\right) \\ &\quad = \mathbf{E}^x \left(t < \tau_D; \int_D \int_0^\infty p_D(s, X_t, y) ds \int_A \nu(y-z) dz dy\right) \\ &\quad = \int_D \int_0^\infty \mathbf{E}^x(t < \tau_D; p_D(s, X_t, y)) ds \int_A \nu(y-z) dz dy \\ &\quad = \int_D \int_0^\infty \int_{\mathbf{R}^d} p_D(t,x,z) p_D(s,z,y) dz ds \int_A \nu(y-z) dz dy. \end{aligned}$$

Now using the semigroup property for p_D (Theorem 1 from [R]) and changing the limits in the second integral, this is equal to

$$\int_D \int_t^\infty p_D(s,x,y) ds \int_A \nu(y-z) dz dy.$$

If $\operatorname{dist}(A, D) = 0$ let us put $A_\epsilon = \{z \in A : \operatorname{dist}(z, D) > \epsilon\}$, $\epsilon > 0$. Then (2.7) holds for A_ϵ. Letting $\epsilon \to 0$ we get (2.7) for A. $\square$

To prove the next generalization we need the following fact.

Lemma 2.6. *For any open nonempty set D, and $D_R = D \cap B(0,R)$,*

$$p_{D_R}(t,x,y) \nearrow p_D(t,x,y)$$

as $R \longrightarrow \infty$, for any $x, y \in D$, $t > 0$.

Proof. We choose R large enough so that x,$y \in D_R$. By the definition of p_D (2.3), $p_D(t,x,y) - p_{D_R}(t,x,y)$ is equal to

$$\mathbf{E}^x(\tau_{D_R} < t; p(t-\tau_{D_R}, X(\tau_{D_R}), y)) - \mathbf{E}^x(\tau_D < t; p(t-\tau_D, X(\tau_D), y)). \tag{2.8}$$

Note that $\tau_{D_R} \le \tau_D$ and

$$\begin{aligned}\mathbf{E}^x(\tau_D < t, &\tau_{D_R} = \tau_D; p(t-\tau_D, X(\tau_D), y)) \\ &= \mathbf{E}^x(\tau_{D_R} < t, \tau_{D_R} = \tau_D; p(t-\tau_{D_R}, X(\tau_{D_R}), y)).\end{aligned}$$

Therefore (2.8) is equal to

$$\begin{aligned}\mathbf{E}^x(\tau_{D_R} < t, \tau_{D_R} < \tau_D; p(t-\tau_{D_R}, X(\tau_{D_R}), y)) \\ - \mathbf{E}^x(\tau_{D_R} < \tau_D < t; p(t-\tau_D, X(\tau_D), y)) \\ \le \mathbf{E}^x(\tau_{D_R} < t,\ \tau_{D_R} < \tau_D; p(t-\tau_{D_R}, X(\tau_{D_R}), y)) \\ = \mathbf{E}^x(\tau_{D_R} < t, X(\tau_{D_R}) \in D; p(t-\tau_{D_R}, X(\tau_{D_R}), y)).\end{aligned}$$

Note also that if $X(\tau_{D_R}) \in D$, then $X(\tau_{D_R}) \in B^c(0,R)$. Thus using Lemma 2.2 the last expression is bounded from the above by

$$\begin{aligned}&\mathbf{E}^x\left(\tau_{D_R} < t, X(\tau_{D_R}) \in D; \frac{c_1(t-\tau_{D_R})e^{m(t-\tau_{D_R})}}{|X(\tau_{D_R}) - y|^{d+\alpha}} e^{-c_2|X(\tau_{D_R})-y|}\right) \\ &\le \frac{c_1 t e^{mt}}{(R-|y|)^{d+\alpha}} e^{-c_2(R-|y|)}.\end{aligned}$$

The last expression tends to 0 as $R \longrightarrow \infty$, thus the lemma is proved. □

Proposition 2.7. *Assume that D is open and nonempty (it may be unbounded) and A is a Borel set such that $A \subset D^c \setminus \partial D$. Also assume that $0 \le t_1 < t_2 < \infty$. Then* (2.7) *holds.*

Proof. Without loss of generality we can assume that $t_1 = 0$. Consider the family of sets $D_R = D \cap B(0,R)$, $R > 0$. These sets are open, bounded and nonempty for large enough R, so we may apply Proposition 2.5 to those sets and $t_2 = t \ge 0$:

$$\mathbf{P}^x(X(\tau_{D_R}) \in A, \tau_{D_R} < t) = \int_{D_R} \int_0^t p_{D_R}(s,x,y)ds \int_A \nu(y-z)dzdy. \tag{2.9}$$

The proof will be completed if we show that

$$\mathbf{P}^x(X(\tau_{D_R}) \in A, \tau_{D_R} < t) \longrightarrow \mathbf{P}^x(X(\tau_D) \in A, \tau_D < t), \tag{2.10}$$

$$p_{D_R}(s,x,y) \nearrow p_D(s,x,y), \tag{2.11}$$

as $R \longrightarrow \infty$.

Lemma 2.6 gives (2.11). We need to show (2.10). We may and do assume that $x \in D_R$. Note that if $X(\tau_{D_R}) \in A$, then $\tau_{D_R} = \tau_D$. Thus

$$\begin{aligned}
&\mathbf{P}^x(X(\tau_D) \in A, \tau_D < t) - \mathbf{P}^x(X(\tau_{D_R}) \in A, \tau_{D_R} < t) \\
&\quad = \mathbf{P}^x(X(\tau_D) \in A, X(\tau_{D_R}) \in D, \tau_{D_R} < \tau_D < t) \\
&\quad \le \mathbf{P}^x(X(\tau_{D_R}) \in D \setminus B(0,R), \tau_{D_R} < t) \\
&\quad = \mathbf{P}^x(X(\tau_{D_R}) \in D \setminus B(0,R), \tau_{D_R} < t, \tau_{B(0,R)} < t) \\
&\quad \le \mathbf{P}^x(\tau_{B(0,R)} < t).
\end{aligned}$$

For $R > 2|x|$ this is bounded from above by

$$\begin{aligned}
&\mathbf{P}^x(\tau_{B(x,R/2)} < t) = \mathbf{P}^0(\tau_{B(0,R/2)} < t) \\
&\quad = 1 - P^0(\tau_{B(0,R/2)} > t) = 1 - \int_{\mathbf{R}^d} p_{B(0,R/2)}(t,0,y)dy.
\end{aligned}$$

By Lemma 2.6 this decreases to $1 - \int_{\mathbf{R}^d} p(t,0,y)dy = 0$. $\square$

The above proposition gives an explicit formula for the joint distribution of $X(\tau_D)$ and τ_D, thus as an easy consequence we have

Corollary 2.8. *Assume that D is an open and nonempty set (it may be unbounded) and A is a Borel set such that $A \subset D^c \setminus \partial D$. Also assume that $0 \le t < \infty$ and B is any Borel set. Then*

$$\begin{aligned}
&\mathbf{P}^x(X(\tau_D) \in A, \tau_D < t, X(t) \in B) \\
&\quad = \int_D \int_0^t p_D(s,x,y) \int_A \nu(y-z)\mathbf{P}^z(X(t-s) \in B)dz\,ds\,dy. \qquad (2.12)
\end{aligned}$$

In the sequel we will need another generalization of the Ikeda-Watanabe formula. Namely we need to change the assumption $A \subset D^c \setminus \partial D$ to the assumption $A \subset D^c$. It is possible to do so, but we have to make some additional regularity assumptions on ∂D.

We say that an open set $D \subset \mathbf{R}^d$ satisfies *the outer cone property* if there exist constants $\eta = \eta(D)$, $R_0 = R_0(D)$ and a cone $C = \{x = (x_1, \ldots, x_d) \in \mathbf{R}^d : 0 < x_d, ||(x_1, \ldots, x_{d-1})|| < \eta x_d\}$ such that for every $Q \in \partial D$, there is a cone C_Q with vertex Q, isometric with C and satisfying $C_Q \cap B(Q, R_0) \subset D^c$.

For such sets we will be able to prove that $P^x(X(\tau_D) \in \partial D; \tau_D < \infty) = 0$, $x \in D$.

At first we need the following auxiliary lemma. We point out that this lemma would be trivial for the symmetric stable process because of the scaling properties of the process. For the relativistic process the proof requires more technical details.

Lemma 2.9. *Let D be an open, nonempty, bounded set satisfying the outer cone property. For $x \in D$ let $r_x = \frac{1}{3}\operatorname{dist}(x, D^c)$ and $B_x = B(x, r_x)$. There exists a constant $p = p(D) > 0$ (not depending on x) such that for any $x \in D$ we have*

$$P^x(X(\tau_{B_x}) \in D^c) > p.$$

Proof. Let $x \in D$. By Lemma 7 from [R] we have

$$E^x(\tau_{B_x}) \ge (1 + mc|D|^{\alpha/d})^{-1} E^x(\tilde{\tau}_{B_x}),$$

where $\tilde{\tau}_{B_x}$ is the exit time for the symmetric α-stable process and c a constant depending only on d and α. It is well known (see e.g. [BK] (2.10)) that $E^x(\tilde{\tau}_{B_x}) =$

$c' r_x^\alpha$, where c' depends only on d and α. Therefore $E^x(\tau_{B_x}) \geq c r_x^\alpha$, where $c = c(D)$ (recall that we omit the dependence on α, d, m in the notation). Let $Q \in \partial D$ be such that $\operatorname{dist}(x, \partial D) = |Q - x|$. We have $C_Q \cap B(Q, R_0) \subset D^c$. By Proposition 2.1 and formula (2.1) we get

(2.13)

$$\begin{aligned} P^x(X(\tau_{B_x}) \in D^c) &= \int_{B_x} G_{B_x}(x,y) \int_{D^c} \nu(y-z)\, dz\, dy \\ &\geq \int_{B_x} G_{B_x}(x,y) \int_{C_Q \cap B(Q,R_0)} c|y-z|^{-d-\alpha} e^{-m^{1/\alpha}|y-z|} \varphi(m^{1/\alpha}|y-z|)\, dz\, dy. \end{aligned}$$

Note that for $y \in B_x$ and $z \in C_Q \cap B(Q, R_0)$ we have $|y - z| \leq \operatorname{diam}(D) + R_0$ and $|y - z| \leq |y - x| + |x - z| \leq 2|x - z|$. We estimate the terms in the integral over $C_Q \cap B(Q, R_0)$ as $|y - z|^{-d-\alpha} \geq c|x - z|^{-d-\alpha}$, $e^{-m^{1/\alpha}|y-z|} \geq c$, $c = c(D)$. By (2.2) $\varphi(\xi) \geq c$ for any $\xi \geq 0$. $\int_{B_x} G_{B_x}(x, y)\, dy = E^x(\tau_{B_x}) \geq c r_x^\alpha$, $c = c(D)$. Therefore using (2.13) we obtain

$$P^x(X(\tau_{B_x}) \in D^c) \geq c r_x^\alpha \int_{C_Q \cap B(Q,R_0)} |x-z|^{-d-\alpha}\, dz, \quad c = c(D). \tag{2.14}$$

We will consider 2 cases, $r_x \geq R_0/2$ and $r_x < R_0/2$.

If $r_x \geq R_0/2$, then the estimate is easy. Note that $r_x^\alpha \geq (R_0/2)^\alpha$ and $|x - z| \leq \operatorname{diam}(D) + R_0$, so the right-hand side of (2.14) is bounded from below by $c(D)$.

If $r_x < R_0/2$ put $C'_Q = C_Q \cap B(Q, R_0) \cap B^c(Q, r_x)$ ($z \in C'_Q$ if $z \in C_Q$ and $r_x \leq |z-Q| \leq R_0$). For $z \in C'_Q$ we have $|x-z| \leq |x-Q|+|Q-z| = 3r_x+|Q-z| \leq 4|Q-z|$. Therefore

$$\int_{C'_Q} |x-z|^{-d-\alpha}\, dz \geq c \int_{C'_Q} |Q-z|^{-d-\alpha}\, dz = c(D) r_x^{-\alpha}.$$

□

Lemma 2.10. *Let D be an open, nonempty set (it may be unbounded) satisfying the outer cone property. Then for any $x \in D$ we have*

$$P^x(X(\tau_D) \in \partial D; \tau_D < \infty) = 0.$$

Proof. At first we note that $|\partial D| = 0$ (the Lebesgue measure). This follows from the fact that almost every point of a measurable set is a density point. Since for $x \in \partial D$ there exists a cone belonging to D^c, the point x is not a density point for ∂D. Therefore $|\partial D| = 0$.

In the next step let us assume that D is bounded. Then we may repeat the proof which is given in [Bo], Lemma 6, for symmetric α-stable processes. Some changes in the proof are needed because we have to prove it for relativistic stable processes which do not have nice scaling properties. Therefore we will repeat the main steps from the proof of [Bo], Lemma 6.

By Theorems 3 and 2 from [R] we have $E^x(\tau_{B(0,R)}) < \infty$ for any $R > 0$ and $x \in B(0, R)$. Since D is bounded it follows that $E^x(\tau_D) < \infty$, hence $P^x(\tau_D < \infty) = 1$, for any $x \in D$. Therefore we will omit $\tau_D < \infty$ and write $P^x(X(\tau_D) \in \partial D)$ instead of $P^x(X(\tau_D) \in \partial D; \tau_D < \infty)$ whenever D is bounded.

For $x \in D$ let $r_x = \frac{1}{3}\operatorname{dist}(x, D^c)$ and $B_x = B(x, r_x)$. By the strong Markov property we have for $x \in D$

$$\begin{aligned} &P^x(X(\tau_D) \in \partial D) \\ &= P^x(X(\tau_{B_x}) \in \partial D) + E^x(P^{X(\tau_{B_x})}(X(\tau_D) \in \partial D); X(\tau_{B_x}) \in D). \end{aligned} \tag{2.15}$$

We denote the two terms on the right-hand side of (2.15) by $p_0(x)$ and $r_0(x)$, respectively. We observe that $p_0(x)$ is the probability of the event that the process X_t jumps directly to ∂D while leaving B_x, and $r_0(x)$ is the probability of a complimentary event that upon leaving B_x it visits D before going to ∂D.

Using (2.15) we can prove inductively that for $k = 0, 1, \ldots$ we have

$$P^x(X(\tau_D) \in \partial D) = p_0(x) + p_1(x) + \ldots + p_k(x) + r_k(x), \quad x \in D, \tag{2.16}$$

with

$$p_{k+1}(x) = E^x(p_k(X(\tau_{B_x})); X(\tau_{B_x}) \in D) \tag{2.17}$$

and

$$r_{k+1}(x) = E^x(r_k(X(\tau_{B_x})); X(\tau_{B_x}) \in D). \tag{2.18}$$

Indeed, it is enough to verify that $r_k = p_{k+1} + r_{k+1}$. We may think of p_k as the probability of the event that the process X_t goes to ∂D after precisely k jumps from one ball B_x to another.

Recall that we assume that D is bounded so we can use Lemma 2.9. By this lemma and (2.18) we get

$$\sup_{x\in D} r_{k+1}(x) \le (1-p) \sup_{x \in D} r_k(x) \le (1-p)^{k+1} \to 0 \quad \text{as} \quad k \to \infty.$$

By (2.16) we obtain $P^x(X(\tau_D) \in \partial D) = \sum_{k=0}^{\infty} p_k(x)$.

Now note that for any $x \in D$, $\operatorname{dist}(B_x, \partial D) > 0$, so we may apply Proposition 2.1. But $|\partial D| = 0$, so this proposition gives that $p_0(x) = 0$ for all $x \in D$. By (2.17) we obtain that $p_k(x) = 0$ for all $x \in D$ and all k. Therefore $P^x(X(\tau_D) \in \partial D) = 0$.

Next let us assume that D is unbounded. Let $D_n = D \cap B(0, n)$, $n = 1, 2, \ldots$. For $x \in D_n$ we have

$$P^x(X(\tau_{D_n}) \in \partial D) = P^x(X(\tau_{D_n}) \in \partial D_n \cap \partial D) + P^x(X(\tau_{D_n}) \in \partial D \setminus \partial D_n).$$

The first term on the right-hand side is 0 because D_n satisfies the outer cone property and is bounded. The second term is 0 by Proposition 2.5 (for $t_1 = 0$, $t_2 = \infty$) and the fact that $|\partial D| = 0$.

Now we will use a very general fact that for each fixed $\omega \in \Omega$ and $t > 0$, we have $\sup_{s\in[0,t]} |X_s(\omega)| < \infty$. Therefore for any fixed $\omega \in \Omega$ such that $\tau_D(\omega) < \infty$, we have $\sup_{s\in[0,\tau_D(\omega)]} X_s(\omega) < \infty$. Hence for any fixed $\omega \in \Omega$ such that $\tau_D(\omega) < \infty$, there exists $n = n(\omega)$ such that $\tau_D(\omega) = \tau_{D_n}(\omega)$. It follows that

$$P^x(X(\tau_D) \in \partial D; \tau_D < \infty) = \lim_{n\to\infty} P^x(X(\tau_{D_n}) \in \partial D) = 0.$$

□

As an immediate conclusion of Corollary 2.8 and Lemma 2.10 we obtain the following corollary.

Corollary 2.11. *Assume that D is an open and nonempty set satisfying the outer cone property (it may be unbounded) and A is a Borel set such that $A \subset D^c$. Also assume that $0 \le t < \infty$ and B is any Borel set. Then (2.12) holds.*

3. Compactness of the Feynman-Kac semigroup

At the beginning of this section we prove the existence and basic properties of the kernel $u(t,x,y)$. The proof is standard and is based on [CZ] (see Section 3.2). Let us denote $P_t f(x) = E^x(f(X_t))$. Using estimates for $p(t,x,y)$ (see Lemma 2.2) it is easy to show that $P_t : L^1(\mathbf{R}^d) \to L^\infty(\mathbf{R}^d)$, $P_t : L^1(\mathbf{R}^d) \to L^1(\mathbf{R}^d)$ and $P_t : L^\infty(\mathbf{R}^d) \to L^\infty(\mathbf{R}^d)$ are bounded operators. By $CB(\mathbf{R}^d)$ we denote the set of all continuous and bounded functions on $\mathbf{R}^d$.

Lemma 3.1. (i) *$T_t(|f|)(x) \le P_t(|f|)(x)$, $t > 0$, $x \in \mathbf{R}^d$, $f : \mathbf{R}^d \to \mathbf{R}$.*

(ii) *For any $t > 0$, $T_t : L^\infty \to CB(\mathbf{R}^d)$.*

(iii) *There exists a density $u(t,x,y)$ for T_t, i.e. $T_t f(x) = \int u(t,x,y) f(y)\,dy$, $t > 0$, $x \in \mathbf{R}^d$, $f \in L^p(\mathbf{R}^d)$ $(1 \le p \le \infty)$. For each fixed $t > 0$, $u(t,x,y)$ is continuous and bounded on $\mathbf{R}^d \times \mathbf{R}^d$.*

(iv) *$u(t,x,y) = u(t,y,x)$, $t > 0$, $x, y \in \mathbf{R}^d$.*

(v) *$u(t,x,y) \le p(t,x,y)$, $t > 0$, $x, y \in \mathbf{R}^d$.*

(vi) *$u(t,x,y) > 0$, $t > 0$, $x, y \in \mathbf{R}^d$.*

Proof. (i) This is clear from (1.1) and our assumption that $V \ge 0$.

(ii) Put $u(x,y) = \int_0^\infty p(s,x,y)\,ds$, $x, y \in \mathbf{R}^d$. By $\mathcal{J}$ we denote the Kato class for the relativistic α-stable process. We say that $q : \mathbf{R}^d \to \mathbf{R}$ belongs to $\mathcal{J}$ if and only if

$$\lim_{r \downarrow 0} [\sup_{x \in \mathbf{R}^d} \int_{|y-x| \le r} u(x,y) |q(y)|\,dy] = 0. \tag{3.1}$$

This definition is motivated by [Z], Theorem 1 (C1). The condition (3.1) implies

$$\lim_{t \downarrow 0} [\sup_{x \in \mathbf{R}^d} E^x \left(\int_0^t |q(X_s)|\,ds \right)] = 0. \tag{3.2}$$

This follows from Lemma 2 in [Z] and proof of Theorem 1, steps 4 and 2 in [Z]. Put $e_q(t) = \exp(\int_0^t q(X_s) ds)$. By a standard argument based on the Khasminskii lemma (see Lemma 3.7 and Proposition 3.8 in [CZ]) (3.2) implies

$$\lim_{t \downarrow 0} [\sup_{x \in \mathbf{R}^d} E^x(e_{|q|}(t))] = 1. \tag{3.3}$$

Now for any $R > 0$, $x \in \mathbf{R}^d$ put $V_R(x) = 1_{B(0,R)}(x) V(x)$. Recall our assumption that $d > \alpha$ and V is locally bounded. Estimates of $u(x,y)$ from Lemma 4 in [R] give that $V_R \in \mathcal{J}$ for any $R > 0$. Set $T_{t,R} f(x) = E^x(e_{-V_R}(t) f(X_t))$, $t > 0$, $R > 0$, $x \in \mathbf{R}^d$. Formula (7) from [R] implies that for each fixed $t > 0$ the kernel $p(t,x,y)$ is bounded and continuous on $\mathbf{R}^d \times \mathbf{R}^d$. It follows that $P_t : L^\infty(\mathbf{R}^d) \to CB(\mathbf{R}^d)$. Using this, formula (3.3) and the same arguments as in the proofs of Propositions 3.11 and 3.12 from [CZ], we obtain that $T_{t,R} : L^\infty(\mathbf{R}^d) \to CB(\mathbf{R}^d)$. We also have

$$|T_t f(x) - T_{t,R} f(x)| = |E^x((e_{-V}(t) - e_{-V_R}(t)) f(X_t))| \le ||f||_\infty P^x(\tau_{B(0,R)} < t).$$

For each fixed $t > 0$, $P^x(\tau_{B(0,R)} < t)$ tends to 0 as R tends to ∞. This implies (ii).

(iii)–(v) By (i) and properties of P_t we obtain $T_t : L^1(\mathbf{R}^d) \to L^\infty(\mathbf{R}^d)$ and $T_t : L^1(\mathbf{R}^d) \to L^1(\mathbf{R}^d)$ are bounded operators. It follows from this and a theorem due to Dunford and Pettis (see [S], Theorem A.1.1, Corollary A.1.2) (cf. also [CZ], page 77) that for each $t > 0$ there exists a measurable (on $\mathbf{R}^d \times \mathbf{R}^d$) density $u(t,x,y)$,

$x, y \in \mathbf{R}^d$ for T_t, i.e.

$$T_t f(x) = \int u(t,x,y) f(y)\, dy, \quad f \in L^1(\mathbf{R}^d),\ t > 0,\ x \in \mathbf{R}^d. \tag{3.4}$$

In fact, by (i) and properties of P_t it is not difficult to show that this formula holds for all $f \in L^p$ $(1 \le p \le \infty)$.

(i) and (1.1) imply that for each fixed $t > 0$ and $x \in \mathbf{R}^d$ we have $0 \le u(t,x,y) \le p(t,x,y)$ for almost all $y \in \mathbf{R}^d$. We may and do assume that these two inequalities hold for all $y \in \mathbf{R}^d$. In particular this gives (v).

By standard arguments (see [CZ], pages 75-76) T_t is symmetric, so for each fixed $t > 0$ (iv) holds for almost all (x,y) according to the Lebesgue measure on $\mathbf{R}^d \times \mathbf{R}^d$.

Put $f_{t,x}(y) = u(t,x,y)$. Fix $t > 0$, $x_0, y_0 \in \mathbf{R}^d$, $r > 0$. Using (iv) (for almost all $(x,y) \in \mathbf{R}^d \times \mathbf{R}^d$) and the semigroup property we get

$$\int_{B(y_0,r)} u(t,x_0,y)\, dy = \int_{B(y_0,r)} T_{t/2} f_{t/2,x_0}(y)\, dy.$$

$f_{t/2,x_0} \in L^\infty(\mathbf{R}^d)$, so (ii) implies that $T_{t/2} f_{t/2,x_0} \in CB(\mathbf{R}^d)$. Therefore we may and do assume that for each fixed $t > 0$ and $x \in \mathbf{R}^d$, $u(t,x,y)$ is continuous as a function of y.

Fixed $t > 0$. For any $x, y \in \mathbf{R}^d$ we have

$$u(t,x,y) = \int\int u(t/3,x,z) u(t/3,z,w) u(t/3,w,y)\, dw\, dz.$$

For any $z, w \in \mathbf{R}^d$, $u(t/3,z,x) \to u(t/3,z,x_0)$ and $u(t/3,w,y) \to u(t/3,w,y_0)$ when $x \to x_0$ and $y \to y_0$. Using the bounded convergence theorem we get (iii). It follows that (iv) holds for all $t > 0$, $x, y \in \mathbf{R}^d$.

(vi) By (1.1) and the fact that V is locally bounded, we obtain that for each fixed $t > 0$, $x \in \mathbf{R}^d$, $u(t,x,y) > 0$ for almost all $y \in \mathbf{R}^d$ according to the Lebesgue measure. We also have $u(t,x,y) = \int u(t/2,x,z) u(t/2,z,y)\, dz$, so (vi) holds for all $t > 0$, $x, y \in \mathbf{R}^d$. □

Now we will prove Theorem 1.1. We define an auxiliary operator.

Definition 3.2. Fix $t > 0$. For any bounded Borel $A \subset \mathbf{R}^d$ let

$$S_A(f)(x) = \int_A u(t,x,y) f(y) dy, \quad f \in L^2(\mathbf{R}^d).$$

Lemma 3.3. *For any fixed $t > 0$ and any bounded Borel $A \subset \mathbf{R}^d$, the operator S_A is compact.*

Proof. It is sufficient to show that this operator is the Hilbert-Schmidt operator. $K(x,y) = 1_A(y) u(t,x,y)$ is the integral kernel of S_A. Using the previous lemma and boundedness of $p(t,x,y)$ (Lemma 2.2) we get

$$\begin{aligned}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} K^2(x,y) dx dy &= \int_{\mathbf{R}^d}\int_A u^2(t,x,y) dy dx \le \int_{\mathbf{R}^d}\int_A p^2(t,x,y) dy dx \\ &\le c_t \int_A \int_{\mathbf{R}^d} p(t,x,y) dy dx = c_t |A| < \infty.\end{aligned}$$

Thus A is the Hilbert-Schmidt operator, so it is compact. □

Lemma 3.4. *Assume that* $\lim_{|x|\to\infty} V(x) = \infty$. *Fix* $t > 0$. *Then we have*

$$\lim_{R\longrightarrow\infty} \sup_{y\in B^c(0,R)} T_t(1_{\mathbf{R}^d})(y) = 0.$$

Proof. Let $M = \inf_{x\in B^c(0,R-1)} V(x)$. Obviously $M \longrightarrow \infty$ as $R \longrightarrow \infty$. Let $\tau = \tau_{B(y,1)}$.

$$\begin{aligned}
\sup_{y\in B^c(0,R)} T_t(1_{\mathbf{R}^d})(y) &\le \sup_{y\in B^c(0,R)} \mathbf{E}^y\left(\exp\left(-\int_0^{\min\{t,\tau\}} V(X_s)ds\right)\right)\\
&= \sup_{y\in B^c(0,R)} \left\{ \mathbf{E}^y\left(\tau > t; \exp\left(-\int_0^t V(X_s)ds\right)\right)\right.\\
&+ \mathbf{E}^y\left(\frac{t}{\sqrt{M}} < \tau < t; \exp\left(-\int_0^\tau V(X_s)ds\right)\right)\\
&\left.+ \mathbf{E}^y\left(\tau < \frac{t}{\sqrt{M}}; \exp\left(-\int_0^\tau V(X_s)ds\right)\right)\right\}\\
&\le e^{-Mt} + e^{-\sqrt{M}t} + \mathbf{P}^0\left(\tau_{B(0,1)} < \frac{t}{\sqrt{M}}\right).
\end{aligned}$$

The lemma follows because the last expression tends to 0 as $R \longrightarrow \infty$. □

Definition 3.5. The ε-net, $\varepsilon > 0$, for the operator $T : L^2(\mathbf{R}^d) \longrightarrow L^2(\mathbf{R}^d)$ is a finite set N_ε of a function from $L^2(\mathbf{R}^d)$, such that for any $f \in L^2(\mathbf{R}^d)$ with $||f||_2 \le 1$ there exists $g \in N_\varepsilon$ such that

$$||T(f) - g||_2 < \varepsilon.$$

It is easy to see that operator is compact iff for every $\varepsilon > 0$, there exists the ε-net for this operator.

Proof of Theorem 1.1. Recall that $V \ge 0$. Assume that $V(x) \to \infty$ as $|x| \to \infty$, and fix $t > 0$ and $\varepsilon > 0$. We will show that there exists the ε-net for T_t. Choose $R > 0$ such that $\sup_{y\in B^c(0,R)} T_t(1_{\mathbf{R}^d})(y) < (\varepsilon/2)^2$. It is possible due to Lemma 3.4. Put $A = B^c(0,R)$. The operator S_{A^c} is compact, so there exists the $(\varepsilon/2)$-net for this operator. Let us denote this net by $N_{\varepsilon/2}$. We will show that $N_{\varepsilon/2}$ is the ε-net for T_t. Fix arbitrary $f \in L^2(\mathbf{R}^d)$, $||f||_2 \le 1$. Put $f_1 = 1_A f$ and $f_2 = 1_{A^c} f$. By the Cauchy-Schwartz inequality we have

$$\begin{aligned}
||T_t(f_1)||_2^2 &= \int_{\mathbf{R}^d}\left(\int_A u(t,x,y)f_1(y)dy\right)^2 dx\\
&\le \int_{\mathbf{R}^d}\left(\int_A u(t,x,y)dy\right)\left(\int_A u(t,x,y)f_1^2(y)dy\right)dx\\
&\le \int_A\int_{\mathbf{R}^d} u(t,x,y)f_1^2(y)dxdy \le \int_A f_1^2(y)T_t(1_{\mathbf{R}^d})(y)dy\\
&\le \sup_{y\in A} T_t(1_{\mathbf{R}^d})(y)\int_A f_1^2(y)dy \le (\varepsilon/2)^2||f_1||_2^2 \le (\varepsilon/2)^2.
\end{aligned}$$

On the other hand we have

$$T_t(f_2)(x) = \int_{\mathbf{R}^d} u(t,x,y)f_2(y)dy = \int_{A^c} u(t,x,y)f(y)dy = S_{A^c}(f)(x).$$

Since $N_{\varepsilon/2}$ is the $(\varepsilon/2)$-net for S_{A^c}, there exists $g \in N_{\varepsilon/2}$ such that $||S_{A^c}(f)-g||_2 < \varepsilon/2$. Now it is sufficient to show that $||T_t(f) - g||_2 < \varepsilon$. Indeed

$$||T_t(f) - g||_2 = ||T_t(f_1 + f_2) - g||_2 \leq ||S_{A_c}(f) - g||_2 + ||T_t(f_1)||_2 \leq \varepsilon.$$

Thus operator T_t is compact.

Now fix $t > 0$ and assume that there exists $N > 0$ and a sequence of disjoint unit balls $B_n = B(x_n, 1)$, such that $V(x) < N$ for any $x \in B_n$, $n \in \mathbb{N}$. We may and do assume there exists $M > 2$ such that $\mathrm{dist}(B_n, B_m) > M$ for any n, $m \in \mathbb{N}$, $n \neq m$. Note that $M > 2$ may be chosen arbitrarily. We will choose appropriate $M > 2$ later in the proof. Now consider the sequence $f_n = 1_{B_n}/(2|B_n|)^{1/2}$. All those functions have norms equal to $1/2$, and

$$\begin{aligned} ||T_t(f_n) - T_t(f_m)||_2^2 = \frac{1}{2|B_1|} \int_{\mathbf{R}^d} \Big[&\mathbf{E}^x \left(X_t \in B_n; e^{-\int_0^t V(X_s)ds} \right) \\ &- \mathbf{E}^x \left(X_t \in B_m; e^{-\int_0^t V(X_s)ds} \right) \Big]^2 dx \\ \geq \frac{1}{2|B_1|} \int_{B(x_n,1/2)} \Big[&\mathbf{E}^x \left(X_t \in B_n; e^{-\int_0^t V(X_s)ds} \right) \\ &- \mathbf{E}^x \left(X_t \in B_m; e^{-\int_0^t V(X_s)ds} \right) \Big]^2 dx. \end{aligned}$$

To show that the operator T_t is not compact, it is sufficient to prove that for $n \neq m$ this norm is greater than a positive constant not depending on n and m. To do this we need to estimate both expected values in the last expression. For $x \in B(x_n, 1/2)$ we have

$$\begin{aligned} \mathbf{E}^x(X_t \in B_n; e^{-\int_0^t V(X_s)ds}) &\geq \mathbf{E}^x(\tau_{B(x,1/2)} > t; e^{-\int_0^t V(X_s)ds}) \\ &\geq e^{-Nt}\mathbf{P}^0(\tau_{B(0,1/2)} > t) = C_1(t)e^{-Nt}. \end{aligned}$$

By Lemma 2.2 for $x \in B(x_n, 1/2)$ and $n \neq m$ we have

$$\begin{aligned} \mathbf{E}^x(X_t \in B_m; e^{-\int_0^t V(X_s)ds}) &\leq \mathbf{P}^x(X_t \in B_m) \\ &= \int_{B_m} p(t,x,y)dy \leq C_2 t e^{mt}|M-2|^{-d-\alpha}. \end{aligned}$$

Now let us choose M large enough so that $\frac{1}{2}C_1(t)e^{-Nt} > C_2te^{mt}|M-2|^{-d-\alpha}$. Then we have

$$||T_t(f_n) - T_t(f_m)||_2^2 \geq \frac{1}{2|B_1|} \int_{B(x_n,1/2)} \frac{1}{4}(C_1(t))^2 e^{-Nt} dx.$$

Thus the points of the sequence $T_t(f_n)$ are separated, so the operator T_t is not compact. □

Now we will show that Condition 1.2 implies IU. Assume that for some open, bounded, nonempty set D and all $t > 0$ Condition 1.2 is satisfied. Recall that ϕ_1 is continuous and strictly positive. Therefore there exists $c_D > 0$ such that for all $x \in D$ we have $\phi_1(x) \geq c_D$. Hence for any $s > 0$ and $x \in \mathbf{R}^d$ we have $T_s(1_D)(x) \leq T_s(\phi_1/c_D)(x) = c_D^{-1}e^{-\lambda_1 s}\phi_1(x)$. By Lemma 3.1 $u(s,x,y) \leq p(s,x,y) \leq c_s$, $s > 0$,

$x, y \in \mathbf{R}^d$. Using the semigroup property we get

$$\begin{aligned} u(t,x,y) &= \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} u(t/3,x,z)u(t/3,z,w)u(t/3,w,y)\,dz\,dy \\ &\le c_{t/3}\int_{\mathbf{R}^d} u(t/3,x,z)\,dz \int_{\mathbf{R}^d} u(t/3,w,y)\,dy = c_{t/3}T_{t/3}(1_{\mathbf{R}^d})(x)T_{t/3}(1_{\mathbf{R}^d})(y) \\ &\le c_{t/3}c_{t/3,D}^2 T_{t/3}(1_D)(x)T_{t/3}(1_D)(y) \le c_{t/3}c_{t/3,D}^2 c_D^{-2}e^{-2\lambda_1 t/3}\phi_1(x)\phi_1(y). \end{aligned} \tag{3.5}$$

This gives (1.2), so the semigroup T_t is IU.

Now we will show that IU implies Condition 1.3. Let $t > 0$ and D be an open, bounded, nonempty set. By (1.3) we have

$$T_t(1_D)(x) = \int_D u(t,x,y)\,dy \ge c_t\phi_1(x)\int_D \phi_1(y)\,dy.$$

Recall that ϕ_1 is bounded. Also by (1.3) we have

$$T_t(1_{B(x,1)})(x) \le c_t\phi_1(x)\int_{B(x,1)} \phi_1(y)\,dy \le c_t||\phi_1||_\infty|B(0,1)|\phi_1(x).$$

This shows that the inequality (1.3) implies Condition 1.3.

4. Estimates of transition probability of the killed process

In this section we prove some estimates of transition probability of the killed process. These estimates will be used in the next section to estimate the probability of "short jumps". On the other hand it seems that these estimates have not been known before and that they are interesting in themselves. The most general result of such type are Theorem 4.2 and Corollaries 4.4 and 4.5.

Lemma 4.1. *Let* $R = B(0,3/2) \setminus \overline{B(0,1)}$ *and* $1 < |x| < 5/4$. *Then*

$$\mathbf{E}^x(\tau_R) \le c\delta_R^{\alpha/2}(x), \tag{4.1}$$

$$\mathbf{P}^x(X(\tau_R) \in B^c(0,3/2)) \le c\delta_R^{\alpha/2}(x). \tag{4.2}$$

Here we recall our notation $\delta_R(x) = \operatorname{dist}(x, \partial R)$.

Proof. By Theorem 3 in [R] (cf. also [CS3]) we get

$$G_R(x,y) \le c\min\left\{\frac{(\delta_R(x)\delta_R(y))^{\alpha/2}}{|x-y|^d}, \frac{\delta_R^{\alpha/2}(x)}{|x-y|^{d-\alpha/2}}\right\}.$$

Therefore

$$\mathbf{E}^x(\tau_R) = \int_R G_R(x,y)dy \le c\int_R \frac{\delta_R^{\alpha/2}(x)}{|x-y|^{d-\alpha/2}}dy \le c\delta_R^{\alpha/2}(x).$$

Let $A = B^c(0,3/2)$. By Corollary 2.11 and Lemma 2.4 we get

$$\begin{aligned} \mathbf{P}^x(X(\tau_R) \in A) &= \int_R G_R(x,y)\int_A \nu(y-z)dzdy \\ &\le c\int_R G_R(x,y)\int_{B^c(y,\delta_A(y))} \frac{1}{|y-z|^{d+\alpha}}dzdy \\ &= c\int_R G_R(x,y)\int_{\delta_A(y)}^\infty \frac{r^{d-1}}{r^{d+\alpha}}drdy = c\int_R G_R(x,y)\frac{1}{\delta_A^\alpha(y)}dy. \end{aligned}$$

Now we divide the last integral into two parts, over sets $R_1 = B(0,5/4) \setminus \overline{B(0,1)}$ and $R_2 = B(0,3/2) \setminus \overline{B(0,5/4)}$. For $y \in R_1$ we have $\delta_A(y) \geq 1/4$, so

$$\int_{R_1} G_R(x,y) \frac{1}{\delta_A^\alpha(y)} dy \leq c \int_{R_1} \frac{\delta_R^{\alpha/2}(x)}{|x-y|^{d-\alpha/2}} 4^\alpha dy \leq c\delta_R^{\alpha/2}(x).$$

For $y \in R_2$ we have $|x-y| \geq 1/4$. Hence

$$\begin{aligned}\int_{R_2} G_R(x,y) \frac{1}{\delta_A^\alpha(y)} dy &\leq c \int_{R_2} \frac{(\delta_R(x)\delta_R(y))^{\alpha/2}}{|x-y|^d} \frac{1}{\delta_A^\alpha(y)} dy \\ &\leq c \int_{R_2} \frac{\delta_R^{\alpha/2}(x)}{\delta_A^{\alpha/2}(y)} 8^d dy \\ &= c\delta_R^{\alpha/2}(x) \int_{5/4}^{3/2} \frac{r^{d-1}}{(3/2-r)^{\alpha/2}} dr \leq c\delta_R^{\alpha/2}(x).\end{aligned}$$

□

The proof of the next theorem contains the main idea of this section.

Theorem 4.2. *Let $D = (\overline{B(0,1)})^c$ and $T > 0$. There exist constants c_T and c such that for any $0 < t \leq T$, $|x| \geq 2$ and $1 < |y| < 5/4$ we have*

$$p_D(t,x,y) \leq c_T e^{-c|x-y|} \delta_D^{\alpha/2}(y).$$

Proof. To show this inequality we will estimate the integral of $p_D(t,y,z)$ over a small ball $B(x,s)$. Then we will take the limit when s tends to 0.

Let $R = B(0,3/2) \setminus \overline{B(0,1)}$ and $s > 0$ such that $B(x,s) \subset B^c(0,3/2)$. We have

$$\begin{aligned}\int_{B(x,s)} p_D(t,y,z)dz &= \mathbf{P}^y(X(t) \in B(x,s), \tau_D > t) \\ &\leq \mathbf{P}^y(\tau_R < t, X(\tau_R) \in D \setminus R, X(t) \in B(x,s)),\end{aligned}$$

and by the strong Markov property the last expression is equal to

$$\mathbf{E}^y(\tau_R < t, X(\tau_R) \in D \setminus R; \mathbf{P}^{X(\tau_R)}(X(t-r) \in B(x,s))|_{r=\tau_R}). \tag{4.3}$$

Now we divide the set $D \setminus R$ into two subsets, $A = B(x,|x-y|/16)$ and $F = D \setminus (A \cup R)$. We have $\mathrm{dist}(F, B(x,s)) \geq |x-y|/32$ for small enough s. Note also that $\mathrm{dist}(A,R) = |x| - 3/2 - |x-y|/16 \geq |x-y|/16$. The last inequality holds because $|x| \geq |x|/8 + |y|/8 + 3/2 \geq |x-y|/8 + 3/2$. Now we are going to estimate the part of (4.3) for the set F. At first note that

$$\begin{aligned}&\mathbf{E}^y(\tau_R < t, X(\tau_R) \in F; \mathbf{P}^{X(\tau_R)}(X(t-r) \in B(x,s))|_{r=\tau_R}) \\ &\qquad = \mathbf{E}^y\left(\tau_R < t, X(\tau_R) \in F; \int_{B(x,s)} p(t-\tau_R, X(\tau_R), z)dz\right).\end{aligned} \tag{4.4}$$

Note also that $X(\tau_R) \in F$, so for $z \in B(x,s)$ we have $|X(\tau_R) - z| \geq |x-y|/32 \geq 1/64$. By Lemma 2.2 this is bounded from above by

$$\begin{aligned}
&\mathbf{E}^y \left(\tau_R < t, X(\tau_R) \in F; \int_{B(x,s)} cte^{mt} e^{-c'|z-X(\tau_R)|} dz \right) \\
&\leq \mathbf{E}^y \left(\tau_R < t, X(\tau_R) \in F; |B(x,s)| cTe^{mT} e^{-c'|x-y|} \right) \\
&\leq c_T |B(x,s)| e^{-c'|x-y|} \mathbf{P}^y (X(\tau_R) \in D \setminus R).
\end{aligned}$$

By the previous lemma we finally obtain

$$\begin{aligned}
&\mathbf{E}^y(\tau_R < t, X(\tau_R) \in F; \mathbf{P}^{X(\tau_R)}(X(t-r) \in B(x,s))|_{r=\tau_R}) \\
&\qquad \leq c_T |B(x,s)| e^{-c'|x-y|} \delta_{B(0,1)}^{\alpha/2}(y).
\end{aligned} \tag{4.5}$$

For the set A by the generalized Ikeda-Watanabe formula (Corollary 2.11) we get

$$\begin{aligned}
&\mathbf{E}^y(\tau_R < t, X(\tau_R) \in A; \mathbf{P}^{X(\tau_R)}(X(t-r) \in B(x,s))|_{r=\tau_R}) \\
&= \mathbf{E}^y(\tau_R < t, X(\tau_R) \in A; X(t) \in B(x,s)) \\
&= \int_R \int_0^t p_R(r,y,z) \int_A \nu(z-w) \mathbf{P}^w (X(t-r) \in B(x,s)) dwdrdz.
\end{aligned} \tag{4.6}$$

Now we will estimate

$$\int_A \nu(z-w) \mathbf{P}^w (X(t-r) \in B(x,s)) dw,$$

for $z \in R$ and $r \in (0,t)$. To do this, we need to divide the set A into two subsets $A_1 = B(x, \max\{2s, q^{1/\alpha}/(64T^{1/\alpha})\})$, where $q = t - r$, and $A_2 = A \setminus A_1$. To make expressions simpler we set $a = \max\{2s, q^{1/\alpha}/(64T^{1/\alpha})\}$ and $b = |x-y|/16$. Note that $q^{1/\alpha}/(64T^{1/\alpha}) < 1/64 < |x-y|/16$, and we may and do assume that $2s < |x-y|/16$. Hence $a < b$. Recall that $\mathrm{dist}(A,R) \geq |x-y|/16$. For $z \in R$ and $w \in A$ we have $|z-w| \geq |x-y|/16 \geq 1/64$. By Lemma 2.3 $\nu(z-w) \leq ce^{-c'|z-w|} \leq ce^{-c'|x-y|}$.

For the set A_2 we have

$$\begin{aligned}
&\int_{A_2} \nu(z-w) \mathbf{P}^w (X(q) \in B(x,s)) dw \\
&\qquad \leq c \int_{A_2} e^{-c'|x-y|} \int_{B(x,s)} p(q,w,u) dudw.
\end{aligned}$$

By Lemma 2.2 this is bounded from above by

$$\begin{aligned}
c \int_{A_2} e^{-c'|x-y|} \int_{B(x,s)} \frac{qe^{mq}}{|w-u|^{d+\alpha}} dudw &\leq c \int_{A_2} e^{-c'|x-y|} \frac{qe^{mT}|B(x,s)|}{\delta_{B(x,s)}^{\alpha+d}(w)} dw \\
&\leq c_T q e^{-c'|x-y|} |B(x,s)| \int_a^b \frac{r^{d-1}}{(r-s)^{d+\alpha}} dr.
\end{aligned}$$

Note that $a \geq 2s$, so this is bounded from above by

$$c_T q e^{-c'|x-y|} |B(x,s)| \int_a^\infty \frac{dr}{r^{1+\alpha}} \leq c_T e^{-c'|x-y|} |B(x,s)| \frac{q}{a^\alpha}.$$

By the definition of a we have $a \geq q^{1/\alpha}/(64T^{1/\alpha})$, hence

$$\int_{A_2} \nu(z-w)\mathbf{P}^w(X(q) \in B(x,s))dw \leq c_T e^{-c'|x-y|}|B(x,s)|. \tag{4.7}$$

We have $A_1 = B(x,a)$. Recall that for $z \in R$ and $w \in A_1 \subset A$ we have $\nu(z-w) \leq ce^{-c'|x-y|}$. Now we will consider two cases. If $a = 2s$, then $|A_1| = c|B(x,s)|$. Hence

$$\begin{aligned} \int_{A_1} \nu(z-w)\mathbf{P}^w(X(q) \in B(x,s))\,dw &\leq \int_{A_1} \nu(z-w)\,dw \\ &\leq ce^{-c'|x-y|}|B(x,s)|. \end{aligned} \tag{4.8}$$

If $a = q^{1/\alpha}/(64T^{1/\alpha})$, then by Lemma 2.2 we get

$$\begin{aligned} &\int_{A_1} \nu(z-w)\mathbf{P}^w(X(q) \in B(x,s))dw \\ &\qquad \leq c\int_{A_1} e^{-c'|x-y|} \int_{B(x,s)} p(q,w,u)dudw \\ &\qquad \leq c\int_{A_1} e^{-c'|x-y|}|B(x,s)|\frac{e^{mq}}{q^{d/\alpha}}dw \\ &\qquad \leq ce^{mT}e^{-c'|x-y|}|B(x,s)|\frac{|A_1|}{q^{d/\alpha}}. \end{aligned}$$

But the Lebesgue measure of A_1 is equal to $c_T q^{d/\alpha}$, so

$$\int_{A_1} \nu(z-w)\mathbf{P}^w(X(q) \in B(x,s))dw \leq c_T e^{-c'|x-y|}|B(x,s)|. \tag{4.9}$$

Substituting (4.7), (4.8) and (4.9) into (4.6) we have

$$\begin{aligned} &\mathbf{E}^y(\tau_R < t, X(\tau_R) \in A; \mathbf{P}^{X(\tau_R)}(X(t-r) \in B(x,s))|_{r=\tau_R}) \\ &\qquad \leq c_T e^{-c'|x-y|}|B(x,s)| \int_R \int_0^\infty p_R(r,y,z)drdz \\ &\qquad = c_T e^{-c'|x-y|}|B(x,s)| \int_R G_R(y,z)dz \\ &\qquad = c_T e^{-c'|x-y|}|B(x,s)|\mathbf{E}^y(\tau_R). \end{aligned}$$

By Lemma 4.1 it follows that

$$\begin{aligned} &\mathbf{E}^y(\tau_R < t, X(\tau_R) \in A; \mathbf{P}^{X(\tau_R)}(X(t-r) \in B(x,s))|_{r=\tau_R}) \\ &\qquad \leq c_T|B(x,s)|e^{-c'|x-y|}\delta^{\alpha/2}_{B(0,1)}(y). \end{aligned} \tag{4.10}$$

From (4.3), (4.5) and (4.10) we finally obtain

$$\frac{1}{|B(x,s)|}\int_{B(x,s)} p_D(t,y,z)dz \leq c_T e^{-c'|x-y|}\delta^{\alpha/2}_{B(0,1)}(y).$$

Letting $s \longrightarrow 0$ we get the assertion of the theorem. □

Corollaries 4.3–4.6 below are simple generalizations and conclusions of Theorem 4.2. It seems that estimates of $p_D(t,x,y)$ presented in these corollaries have not been known before, and they may find some interesting applications.

Almost the same proof as the proof of Theorem 4.2 leads to a slightly more general fact.

Corollary 4.3. *Let $D = (\overline{B(0,\varepsilon)})^c$ and $T > 0$. There exist constants $c_{T,\varepsilon}$ and c_ε such that for any $0 < t \le T$, $|x| \ge 2\varepsilon$ and $\varepsilon < |y| < 5\varepsilon/4$ we have*

$$p_D(t,x,y) \le c_{T,\varepsilon} e^{-c_\varepsilon |x-y|} \delta_D^{\alpha/2}(y).$$

Corollary 4.4. *Let $T > 0$ and let $D \subset \mathbf{R}^d$ be an open set satysfying the outer ball condition in point $z \in \partial D$, that is, there exists a ball $B(w,\varepsilon) \subset D^c$, $\varepsilon > 0$, such that $\partial D \cap \partial B(w,\varepsilon) = \{z\}$. Then there exist constants $c_{T,\varepsilon}$ and c_ε such that for any $0 < t \le T$, x, $y \in D$ and $|x - w| > 2\varepsilon$ we have*

$$p_D(t,x,y) \le \frac{c_{T,\varepsilon}}{|x-y|^{d+\alpha}} e^{-c_\varepsilon |x-y|} |z-y|^{\alpha/2}.$$

Proof. We will consider two cases. First let $|z-y| > \varepsilon/4$. Then the assertion follows from Lemma 2.2. Now assume that $|z-y| < \varepsilon/4$. Then $|y-w| < 5\varepsilon/4$. $B(w,\varepsilon)$ is a subset of D^c, thus $p_D(t,x,y) \le p_{(\overline{B(w,\varepsilon)})^c}(t,x,y)$ for any x,$y \in \mathbf{R}^d$, and any $t > 0$. By the previous corollary

$$p_{(\overline{B(w,\varepsilon)})^c}(t,x,y) \le c_{T,\varepsilon} e^{-c_\varepsilon |x-y|} \delta_{B(w,\varepsilon)}^{\alpha/2}(y),$$

for any $x \notin B(w,2\varepsilon)$. Besides $\delta_{B(w,\varepsilon)}(y) \le |z-y|$ and $|x-y| \ge 3\varepsilon/4$, thus

$$p_D(t,x,y) \le p_{(\overline{B(w,\varepsilon)})^c}(t,x,y) \le \frac{c_{T,\varepsilon}}{|x-y|^{d+\alpha}} e^{-c_\varepsilon |x-y|} |z-y|^{\alpha/2}.$$

□

Corollary 4.5. *Let $T > 0$ and let D be a set with a uniform outer ball condition, i.e. there exists $\varepsilon > 0$, such that for any $z \in \partial D$ there exists $B(w,\varepsilon) \subset D^c$, such that $\partial B(w,\varepsilon) \cap \partial D = \{z\}$. Then there exist constants $c_{T,\varepsilon}$ and c_ε such that for any x, $y \in D$, $\delta_D(x) > \varepsilon$ and $0 < t \le T$ we have*

$$p_D(t,x,y) \le \frac{c_{T,\varepsilon}}{|x-y|^{d+\alpha}} e^{-c_\varepsilon |x-y|} \delta_D^{\alpha/2}(y).$$

Corollary 4.6. *Let $R \ge 1$, $D = (\overline{B(0,R)})^c$ and $0 < T < \infty$. There exist constants c_T and c such that for any $0 < t \le T$, $|x| \ge R+1$ and $R < |y| \le R + 1/4$ we have*

$$p_D(t,x,y) \le c_T e^{-c|x-y|} \delta_D^{\alpha/2}(y).$$

As an application of the above corollary we prove the following proposition, which will be very important in the proof of the main theorem.

Proposition 4.7. *Let $R \ge 1$, $D = (\overline{B(0,R)})^c$ and $0 < T < \infty$. There exists a constant c_T such that for any $0 \le t_1 < t_2 \le T$ and $|x| > R+1$ we have*

$$\mathbf{P}^x(t_1 < \tau_D < t_2) \le c_T (t_2 - t_1).$$

Let us emphasize that c_T does not depend on R.

Proof. By the generalized Ikeda-Watanabe formula (Corollary 2.11)

$$\mathbf{P}^x(t_1 < \tau_D < t_2) = \int_D \int_{t_1}^{t_2} p_D(s,x,y) ds \int_{D^c} \nu(y-z) dz dy.$$

Now we divide the set D into two parts, $A = B^c(0, R+1/4)$ and $B = D \setminus A$. If $y \in A$ and $z \in D^c$, then $|z-y| \ge 1/4$. Hence by Lemma 2.3 $\nu(y-z) \le c e^{-c'|y-z|}$.

It follows that

$$\begin{aligned}(4.11)\qquad &\int_A \int_{t_1}^{t_2} p_D(s,x,y)ds \int_{D^c} \nu(y-z)dzdy \\ &\qquad \le c \int_A \int_{t_1}^{t_2} p_D(s,x,y)ds \int_{D^c} e^{-c'|y-z|}dzdy.\end{aligned}$$

We have $\int_{D^c} e^{-c'|y-z|}dz \le \int_{\mathbf{R}^d} e^{-c'|y-z|}dz < \infty$. Therefore (4.11) is bounded from above by

$$c\int_{t_1}^{t_2}\int_A p_D(s,x,y)dyds = c\int_{t_1}^{t_2} \mathbf{P}^x(\tau_D > s)ds \le c(t_2-t_1).$$

Let $\delta(y) = \delta_{B(0,R)}(y)$. By Lemma 2.4 we have

$$\begin{aligned}&\int_B \int_{t_1}^{t_2} p_D(s,x,y)ds \int_{D^c} \nu(y-z)dzdy \\ &\qquad \le c\int_B \int_{t_1}^{t_2} p_D(s,x,y)ds \int_{B^c(y,\delta(y))} \frac{1}{|y-z|^{d+\alpha}}dzdy \\ &\qquad = c\int_B \int_{t_1}^{t_2} p_D(s,x,y)ds \int_{\delta(y)}^{\infty} \frac{r^{d-1}}{r^{d+\alpha}}drdy \\ &\qquad = c\int_B \int_{t_1}^{t_2} p_D(s,x,y)ds \frac{1}{\delta^{\alpha}(y)}dy,\end{aligned}$$

and by Corollary 4.6 this is bounded from above by

$$c_T(t_2-t_1)\int_B e^{-c|x-y|}\frac{dy}{\delta^{\alpha/2}(y)}.$$

It is sufficient to show that this integral is bounded by a constant not depending on R. We will prove this for $d \ge 3$. For the dimension $d = 1$ the proof is very easy and for $d = 2$ the proof is similar to the proof for $d \ge 3$.

We may and do assume that $x = (0,\dots,0,R+1)$. We introduce spherical coordinates in $\mathbf{R}^d$ $(r,\varphi_1,\varphi_2,\dots,\varphi_{d-1})$, $r \in [0,\infty)$, $\varphi_1, \dots, \varphi_{d-2} \in [0,\pi]$, $\varphi_{d-1} \in [0,2\pi)$, with center in 0 and principal axis $0x$.

We divide the set B into 2 parts:

$$\begin{aligned}B_1 &= \{(r,\varphi_1,\varphi_2,\dots,\varphi_{d-1}) : \ r \in (R,R+1/4),\ \varphi_1 \in [0,1/R]\},\\ B_2 &= \{(r,\varphi_1,\varphi_2,\dots,\varphi_{d-1}) : \ r \in (R,R+1/4),\ \varphi_1 \in (1/R,\pi]\}.\end{aligned}$$

We have

$$\begin{aligned}\int_{B_1} e^{-c|x-y|}\frac{dy}{\delta^{\alpha/2}(y)} &\le c\int_0^{1/R}\int_R^{R+1/4} \frac{r^{d-1}}{(r-R)^{\alpha/2}} \sin^{d-2}\varphi_1 \, drd\varphi_1 \\ &\le cR^{d-1}\int_0^{1/R}\int_0^{1/4} \frac{1}{r^{\alpha/2}}\varphi_1^{d-2} drd\varphi_1 \\ &\le cR^{d-1}(1/R)^{d-1} = c.\end{aligned}$$

Now we estimate the integral over B_2. Let $y \in B_2$ and put $y_0 = yR/|y|$ and $x_0 = xR/|x| = (0,0,\dots,R)$. Note that $|x-y| \ge |x_0-y|$ and $|x_0-y| \ge |x_0-y_0|-1/4$. Let $y = (r,\varphi_1,\dots,\varphi_{d-1})$ in spherical coordinates. Since $y \in B_2$ we have $\varphi_1 > 1/R$.

It follows that $|x_0 - y_0| \geq 2R\sin(1/2R) \geq 2/\pi > 1/2$. Hence $|x_0 - y| \geq |x_0 - y_0|/2$. This gives

$$|x-y| \geq |x_0 - y_0|/2 = R\sin(\varphi_1/2) \geq R\varphi_1/\pi = CR\varphi_1.$$

We also have $e^{-c|x-y|} \leq c'|x-y|^{-d}$. Therefore $e^{-c|x-y|} \leq c'(CR\varphi_1)^{-d}$. It follows that

$$\begin{aligned}\int_{B_2} e^{-c|x-y|}\frac{dy}{\delta^{\alpha/2}(y)} &\leq c'' \int_{1/R}^{\pi}\int_R^{R+1/4} \frac{r^{d-1}}{(r-R)^{\alpha/2}}\frac{\sin^{d-2}\varphi_1}{(R\varphi_1)^d} dr d\varphi_1 \\ &\leq cR^{d-1}\int_0^{1/4}\frac{1}{r^{\alpha/2}}dr\int_{1/R}^{\pi}\frac{\varphi_1^{d-2}}{(R\varphi_1)^d}d\varphi_1 \\ &= cR^{-1}\int_{1/R}^{\pi}\varphi_1^{-2}d\varphi_1 = c.\end{aligned}$$

□

5. Intrinsic ultracontractivity

In this section we prove the main result of this paper, Theorem 1.6. In this section we fix the potential $V \in \mathcal{V}$.

To prove that the semigroup T_t is IU we will use Condition 1.2 for $D = B(0,1)$. To show that the semigroup T_t is not IU we will use Condition 1.3. At first we prove two auxiliary lemmas, which will be used to estimate $T_t(1_D)(x)$.

Lemma 5.1. *Let $0 \leq t_1 < t_2 \leq t < \infty$, $x \in \mathbf{R}^d$, $|x| \geq 3$, $D = B(0,1)$ and $B = B(x,1)$. Then*

$$\mathbf{P}^x(X(\tau_B) \in D/2, t_1 < \tau_B < t_2) \geq \frac{c_t(t_2 - t_1)}{|x|^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}|x|}.$$

Proof. By Proposition 2.5 we have

$$\begin{aligned}&\mathbf{P}^x(X(\tau_B) \in D/2, t_1 < \tau_B < t_2) \\ &\qquad = \int_B \int_{t_1}^{t_2} p_B(s,x,y)ds \int_{D/2} \nu(y-z)dzdy.\end{aligned}$$

By Lemma 2.3 this is bounded from below by

$$\begin{aligned}&\int_B\int_{t_1}^{t_2} p_B(s,x,y)ds|D/2|\frac{c}{(|x|+1)^{\frac{d+\alpha+1}{2}}}e^{-m^{1/\alpha}(|x|+1)}dy \\ &\geq \frac{c}{|x|^{\frac{d+\alpha+1}{2}}}e^{-m^{1/\alpha}|x|}\int_{t_1}^{t_2}\int_B p_B(s,x,y)dyds \\ &= \frac{c}{|x|^{\frac{d+\alpha+1}{2}}}e^{-m^{1/\alpha}|x|}\int_{t_1}^{t_2}\mathbf{P}^x(\tau_B > s)ds \\ &\geq \frac{c(t_2-t_1)}{|x|^{\frac{d+\alpha+1}{2}}}e^{-m^{1/\alpha}|x|}\mathbf{P}^x(\tau_B > t)\end{aligned}$$

and $P^x(\tau_B > t) = c_t$. Here we recall our convention that constants c may change their value from one use to the next. □

Lemma 5.2. *For any $r \geq 0$ we have*

$$\sum_{n=1}^{\infty} \frac{e^{-r/n}}{n(n+1)} \geq \frac{e^{-1}}{r+1}.$$

Proof. We have

$$\sum_{n=1}^{\infty} \frac{e^{-r/n}}{n(n+1)} \geq \sum_{n=[r]+1}^{\infty} \frac{e^{-1}}{n(n+1)} = \frac{e^{-1}}{[r]+1} \geq \frac{e^{-1}}{r+1}.$$

□

Now we will show some lemmas needed to estimate $T_t(1_{\mathbf{R}^d})(x)$ from above. This is the most difficult part of the paper. Roughly speaking, the main idea is to divide $\mathbf{R}^d$ into appropriate rings, and estimate the probability of jumps between these rings. To shorten notation, from now on we will assume that n, k, l, $N \in \mathbb{N}$. We will use the following notation:

- $2 \leq n_0 \in N$ will be chosen later,
- $R_n = \overline{B(0,n)} \setminus \overline{B(0,n-1)}$ for any $n \geq n_0 + 2$,
- $R_{n_0} = \overline{B(0,n_0)}$, $R_{n_0+1} = B(0, n_0+1)$,
- $A_n = (\overline{B(0,n-2)})^c$ for any $n \geq n_0 + 2$,
- $A_{n_0} = \mathbf{R}^d$, $A_{n_0+1} = \mathbf{R}^d$,
- $\tau_n = \tau_{A_n} = \inf\{t \geq 0 : X_t \notin A_n\}$, $n \geq n_0$,
- $\sigma_n = \sigma_{R_n} = \inf\{t \geq 0 : X_t \in R_n\}$, $n \geq n_0$.

Let us point out that for $n \geq n_0 + 2$ sets R_n are "rings" in $\mathbf{R}^d$ and R_{n_0}, R_{n_0+1} are balls. Note also that $\tau_{n_0} = \infty$, $\tau_{n_0+1} = \infty$.

First we need to estimate a volume of the intersection of two balls.

Lemma 5.3. *Let $0 < l \leq n$, $k > 0$. Consider two balls $B(x,n)$ and $B(y, k+l)$ with $|x-y| = n+k$. The volume of the intersection of these balls (denoted by I) is less than $cl^{\frac{d+1}{2}}(\min\{2k+l, 2n\})^{\frac{d-1}{2}}$.*

Proof. Let $d > 1$. Let z be any point belonging to the intersection of spheres with centers at x and y, and radius n and $k+l$. Let w be the orthogonal projection of z on the line containing x and y. Then $k < |w-y| < k+l$. Therefore

$$|z-w|^2 = (k+l)^2 - |w-y|^2 < (k+l)^2 - k^2 = 2kl + l^2 = l(2k+l).$$

Analogical argument can be applied to n, thus we have

$$|z-w|^2 < n^2 - (n-l)^2 = 2nl - l^2 < 2nl.$$

Therefore we get

$$|I| \leq cl|z-w|^{d-1} \leq cl^{\frac{d+1}{2}}(\min\{2k+l, 2n\})^{\frac{d-1}{2}}.$$

Let $d = 1$. In this case the lemma obviously holds, because the intersection of two balls (intervals) has measure equal to l. □

Now we estimate an integral of the Levy measure.

Lemma 5.4. *Let $|x| \geq N \geq n+1$ and $n \geq n_0$. Then we have*

$$\int_{R_n} \nu(x-y)dy \leq \frac{c(\min\{N-n, n\})^{\frac{d-1}{2}}}{(N-n)^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}(N-n)}.$$

Proof. By Lemma 2.3

$$\begin{aligned}\int_{R_n} \nu(x-y)dy &\le \int_{R_n} \frac{c}{|x-y|^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}|x-y|}\, dy \\ &\le \int_{B(0,n)} \frac{c}{|x-y|^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}|x-y|}\, dy =: J.\end{aligned}$$

Let us point out that the last inequality holds both when $n \ge n_0 + 2$ (and R_n are "rings") and when $n = n_0$ or $n = n_0 + 1$ (and R_n are balls).

It is sufficient to consider only the case $x = (0, 0, ..., 0, N)$. Let $N - n = k$. Consider a sequence of balls $B(x, k + l)$ for $1 \le l \le n$. Then

$$\begin{aligned}B(0,n) = {} & \underbrace{(B(0,n) \setminus B(x,k+n))}_{=A} \\ & \cup \bigcup_{0<l\le n} \underbrace{\{(B(0,n) \cap B(x,k+l)) \setminus B(x,k+l-1)\}}_{=A_l}.\end{aligned}$$

In addition we denote $E_l = B(0,n) \cap B(x,k+l)$. We will divide an integral J into integrals over sets A and A_l. At first we estimate an integral over A:

$$\begin{aligned}\int_A \frac{c}{|x-y|^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}|x-y|}\, dy &\le \frac{c|A|}{(k+n)^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}(k+n)} \\ &\le \frac{cn^d}{k^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}(k+n)} \le \frac{c}{k^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}k}.\end{aligned}$$

In the last inequality we use the fact that there exists c (depending on m, α and d) such that for any $n \in \mathbb{N}$ we have $n^d e^{-m^{1/\alpha}n} \le c$. In the sequel we will use similar inequalities without further comments. For A_l we have

$$\begin{aligned}\int_{A_l} \frac{c}{|x-y|^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}|x-y|}\, dy &\le \frac{c|A_l|}{(k+l-1)^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}(k+l-1)} \\ &\le \frac{c|E_l|}{k^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}(k+l)}.\end{aligned}$$

Now we will consider two cases. At first assume that $k \ge n$. Using the previous lemma we obtain

$$\begin{aligned}J &\le \frac{c}{k^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}k} + \sum_{1\le l\le n} \frac{cl^{\frac{d+1}{2}} (\min\{2k+l, 2n\})^{\frac{d-1}{2}}}{k^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}(k+l)} \\ &\le \frac{c}{k^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}k} + \frac{c(\min\{3k, 3n\})^{\frac{d-1}{2}}}{k^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}k} \sum_{1\le l\le n} l^{\frac{d+1}{2}} e^{-m^{1/\alpha}l} \\ &\le \frac{c(\min\{k,n\})^{\frac{d-1}{2}}}{k^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}k}.\end{aligned}$$

In the second case ($k < n$) we have

$$\begin{aligned} J &\le \frac{c}{k^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha} k} + \sum_{1 \le l \le n} \frac{c l^{\frac{d+1}{2}} (\min\{2k+l, 2n\})^{\frac{d-1}{2}}}{k^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}(k+l)} \\ &\le \frac{c}{k^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha} k} + \sum_{1 \le l \le k} \frac{c l^{\frac{d+1}{2}} (\min\{3k, 3n\})^{\frac{d-1}{2}}}{k^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}(k+l)} \\ &+ \sum_{k < l \le n} \frac{c l^{\frac{d+1}{2}} (3l)^{\frac{d-1}{2}}}{k^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}(k+l)} \le \frac{c (\min\{k, n\})^{\frac{d-1}{2}}}{k^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha} k}. \end{aligned}$$

□

Lemma 5.5. *For $N - 1 < |x| \le N$, $n_0 \le n \le N - 2$ and $0 \le t_1 < t_2 \le T < \infty$ we have*

$$\begin{aligned} &\mathbf{P}^x(X(\tau_N) \in R_n, t_1 < \tau_N < t_2) \\ &\qquad \le C_T (t_2 - t_1) \frac{(\min\{N-n, n\})^{\frac{d-1}{2}}}{(N-n)^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}(N-n)}. \end{aligned}$$

Proof. Recall that $\tau_N = \tau_{A_N}$. For $n = N - 2$ we have

$$\mathbf{P}^x(X(\tau_N) \in R_n, t_1 < \tau_N < t_2) \le \mathbf{P}^x(t_1 < \tau_N < t_2) \le c_T (t_2 - t_1),$$

by Proposition 4.7.

Let $n < N - 2$. By Proposition 2.7 (the generalization of the Ikeda-Watanabe formula) we have

$$\begin{aligned} &\mathbf{P}^x(X(\tau_N) \in R_n, t_1 < \tau_N < t_2) \\ &\qquad = \int_{A_N} \int_{t_1}^{t_2} p_{A_N}(s, x, y) ds \int_{R_n} \nu(y - z) dz dy. \end{aligned} \tag{5.1}$$

Note that for $y \in A_N$ we have $|y| > N - 2$. By Lemma 5.4 for $y \in A_N$ we have

$$\int_{R_n} \nu(y - z) dz \le \frac{c \min\{N - 2 - n, n\}^{\frac{d-1}{2}}}{(N - 2 - n)^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}(N-2-n)}.$$

Therefore (5.1) is bounded from the above by

$$\frac{c(\min\{N-n, n\})^{\frac{d-1}{2}}}{(N-n)^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}(N-n)} \int_{t_1}^{t_2} \int_{A_N} p_{A_N}(s, x, y) dy ds.$$

We have $\int_{A_N} p_{A_N}(s, x, y) dy = \mathbf{P}^x(\tau_{A_N} > s) \le 1$, and the lemma follows. □

Lemma 5.6. *For any $r > 0$ we have*

$$\sum_{n=1}^{\infty} \frac{e^{-\frac{r}{n+1}}}{n(n+1)} \le \frac{5}{r}.$$

Proof. We have

$$\sum_{n=1}^{\infty} \frac{e^{-\frac{r}{n+1}}}{n(n+1)} = \frac{e^{-r/2}}{2} + \sum_{n=2}^{\infty} \frac{e^{-\frac{r}{n+1}}}{n(n+1)}.$$

Note that $1/(2e^{r/2}) \le 1/r$. We also have

$$\sum_{n=2}^{\infty} \frac{e^{-\frac{r}{n+1}}}{n(n+1)} \le \int_2^{\infty} \frac{e^{-\frac{r}{s+1}}}{(s-1)s} ds \le 2 \int_2^{\infty} \frac{e^{-\frac{r}{2s}}}{s^2} ds = \frac{4}{r} \int_0^{r/4} e^{-u} du \le \frac{4}{r}.$$

□

Lemma 5.7. *For $N-1 < |x| \le N$, $n_0 \le k \le N-2$, $t > 0$ and $a > 0$ we have*

$$\begin{aligned} &\mathbf{E}^x(\tau_N < t, X(\tau_N) \in R_k; e^{-\tau_N a}) \\ &\le \mathcal{C}_1(t) \frac{(\min\{N-k,k\})^{\frac{d-1}{2}}}{(N-k)^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}(N-k)} \frac{1}{a}. \end{aligned} \tag{5.2}$$

Proof. Using last two lemmas we obtain

$$\begin{aligned} &\mathbf{E}^x(\tau_N < t, X(\tau_N) \in R_k; e^{-\tau_N a}) \\ &\quad = \sum_{i=1}^{\infty} \mathbf{E}^x \left(\frac{t}{i+1} \le \tau_N < \frac{t}{i}, X(\tau_N) \in R_k; e^{-\tau_N a} \right) \\ &\quad \le \sum_{i=1}^{\infty} e^{\frac{-ta}{i+1}} \mathbf{P}^x \left(\frac{t}{i+1} \le \tau_N < \frac{t}{i}, X(\tau_N) \in R_k \right) \\ &\quad = \sum_{i=1}^{\infty} e^{\frac{-ta}{i+1}} C_t \frac{t}{i(i+1)} \frac{(\min\{N-k,k\})^{\frac{d-1}{2}}}{(N-k)^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}(N-k)} \\ &\quad \le \mathcal{C}_1(t) \frac{(\min\{N-k,k\})^{\frac{d-1}{2}}}{(N-k)^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}(N-k)} \frac{1}{a}. \end{aligned}$$

Of course the constant $C_1(t)$ also depends on α, d and m. □

Now we define events, sequences of jumps, which will help to estimate $T_t(1_{\mathbf{R}^d})$. This notation is very essential in the sequel. Roughly speaking the main idea of the paper is to estimate $T_t(1_{\mathbf{R}^d})$ by estimating the appropriate jumps of the process between rings. This idea comes from the paper [BK].

For $k \ge n_0$, $n \ge k+2$ and $t > 0$ we define

- $S(n,k,1,t) = \{X(\tau_n) \in R_k, \sigma_k < t\}$,
- $S(n,k,l,t) = \bigcup_{p=k+2}^{n-2} S(n,p,l-1,t) \cap S(p,k,1,t)$ for any $l \ge 2$,
- $R(n,k,l,t) = S(n,k,l,t) \cap \{\tau_k > t\}$.

When $n \ge k+2$, $n-2 < k+2$ and $l \ge 2$, one should understand that $S(n,k,l,t) = \emptyset$ and $R(n,k,l,t) = \emptyset$. Note also that for fixed n and k the events $S(n,k,l,t)$ are empty for large enough l.

$S(n,k,1,t)$ is the event that the process while leaving A_n jumps directly to R_k and $\sigma_k = \inf\{t \ge 0 : X_t \in R_k\}$, the entry time to R_k, is smaller than t. Note that if $X(\tau_n) \in R_k$, then $\sigma_k = \tau_n$.

$S(n,p,1,t) \cap S(p,k,1,t)$ for $n-2 \ge p \ge k+2$ is the event that the process while leaving A_n jumps directly to R_p and then while leaving A_p jumps directly to R_k and $\sigma_k < t$, $\sigma_p < t$. $S(n,k,2,t)$ for $n-2 \ge k+2$ ($l=2$) is the sum of $S(n,p,1,t) \cap S(p,k,1,t)$ for all p between $k+2$ and $n-2$. Roughly one may think of $S(n,k,2,t)$ as the event that the process goes from A_n to R_k in 2 "appropriate jumps" (and $\sigma_k < t$).

Similarly, by induction we define $S(n,k,l,t)$. One may think of $S(n,k,l,t)$ as the event that the process goes from A_n to R_k in l "appropriate jumps" (and $\sigma_k < t$).

$R(n,k,l,t)$ is the event that the process goes from A_n to R_k in l "appropriate jumps", $\sigma_k < t$, and that the process remains in A_k from σ_k to t.

Note that "jumps" are defined so that when the process jumps to $R_p = \overline{B(0,p)} \setminus B(0,p-1)$, then the next jump is from $A_p = (\overline{B(0,p-2)})^c$ (not from $(\overline{B(0,p-1)})^c$).

This is done for technical reasons. It would be difficult to estimate the probability of jumps from R_p to $(\overline{B(0,p-1)})^c$. The same method was used in [BK].

Let $t > 0$, $n \geq 2n_0 + 4$ and $x \in R_n$. We take $n \geq 2n_0 + 4$ so that $[n/2] \geq n_0 + 2$. Note also that $n - [n/2] \geq 2$. For such t, n and x we have

$$\begin{aligned} T_t(1_{\mathbf{R}^d})(x) = \mathbf{E}^x(e^{-\int_0^t V(X_s)ds}) &\leq \mathbf{E}^x(\tau_{[n/2]} > t; e^{-\int_0^t V(X_s)ds}) \\ &+ \sum_{k=n_0+2}^{[n/2]} \sum_{l=1}^{\infty} \mathbf{E}^x(R(n,k,l,t); e^{-\int_0^t V(X_s)ds}) \\ &+ \sum_{k=n_0}^{n_0+1} \sum_{l=1}^{\infty} \mathbf{E}^x(R(n,k,l,t); e^{-\int_0^t V(X_s)ds}). \end{aligned} \tag{5.3}$$

The terms under the sum $\sum_{k=n_0+2}^{[n/2]} \sum_{l=1}^{\infty}$ correspond to events that the process will make l "appropriate jumps" from A_n to R_k and then remain in A_k up to time t. The terms under the sum $\sum_{k=n_0}^{n_0+1} \sum_{l=1}^{\infty}$ correspond to events that the process will make l "appropriate jumps" from A_n to R_k and here we do not control the behaviour of the process after σ_k ($\tau_{n_0} = \infty$, $\tau_{n_0+1} = \infty$). Roughly speaking these terms for $k = n_0$ and $k = n_0 + 1$ appear separately because we can control the expression of the type $\mathbf{E}^x(S(n,k,l,t), e^{-\frac{1}{2}\int_0^{\sigma_k} V(X_s)ds})$ only when k is big enough (see Lemma 5.9). We have 2 parameters $k = n_0$ and $k = n_0+1$ for technical reasons (because of our definition of jumps).

Now we focus on estimating the single terms from the above sums. For $x \in R_n$, $k \geq n_0 + 2$, $n \geq k + 2$, $l \geq 1$ and $t > 0$ we have

$$\begin{aligned} &\mathbf{E}^x\Big(R(n,k,l,t); e^{-\int_0^t V(X_s)ds}\Big) \\ &\qquad \leq \mathbf{E}^x\left(R(n,k,l,t); e^{-\frac{1}{2}\int_0^{\sigma_k} V(X_s)ds} e^{-\frac{1}{2}\int_0^t V(X_s)ds}\right) \\ &\qquad \leq \mathbf{E}^x\left(S(n,k,l,t), \tau_k > t; e^{-\frac{1}{2}\int_0^{\sigma_k} V(X_s)ds} e^{-\frac{t}{2}L(k-2)}\right) \\ &\qquad \leq e^{-\frac{t}{2}L(k-2)} \mathbf{E}^x\left(S(n,k,l,t); e^{-\frac{1}{2}\int_0^{\sigma_k} V(X_s)ds}\right). \end{aligned} \tag{5.4}$$

By similar arguments we get for $k = n_0$ and $k = n_0 + 1$

$$\mathbf{E}^x\Big(R(n,k,l,t); e^{-\int_0^t V(X_s)ds}\Big) \leq \mathbf{E}^x\left(S(n,k,l,t); e^{-\frac{1}{2}\int_0^{\sigma_k} V(X_s)ds}\right).$$

Now we need the following auxiliary fact.

Lemma 5.8. *Let $n > k > l > 0$. We have*

$$W = \frac{\min\{n-k,k\}}{n-k} \frac{\min\{k-l,l\}}{k-l} \leq \mathcal{C}_2 \frac{\min\{n-l,l\}}{n-l}.$$

Proof. One can easily see that the following inequality holds:

$$(n-k)(k-l) \geq \frac{1}{2}(n-l)\min\{n-k,k-l\}. \tag{5.5}$$

Thus

$$W \leq 2\frac{\min\{n-k,k\}\min\{k-l,l\}}{\min\{n-k,k-l\}} \frac{1}{(n-l)}.$$

Hence it is sufficient to show that

$$Y = \frac{\min\{n-k,k\}\min\{k-l,l\}}{\min\{n-k,k-l\}} \leq c\min\{n-l,l\}.$$

Now we have three cases:

- $n-k \le k-l$,
$$Y = \min\{k-l, l\} \le \min\{n-l, l\},$$
- $n-k > k-l$ and $k-l \le l$,
$$Y = \min\{n-k, k\} \le \min\{n-l, 2l\} \le 2\min\{n-l, l\},$$
- $n-k > k-l$ and $k-l > l$,
$$Y = \frac{\min\{n-k, k\} l}{k-l} \le \frac{kl}{k/2} = 2l$$
and
$$Y = \frac{\min\{n-k, k\} l}{k-l} \le \frac{(n-k)(k-l)}{k-l} = n-k \le n-l.$$

□

The following lemma is crucial in our considerations. Roughly speaking we estimate $e^{-\int_0^t V(X_s)ds}$ depending on the number of "appropriate jumps" the process made. The idea of the proof of this lemma is taken from [BK], Lemma 4.5.

In this lemma we will not use our convention that constants may change their value from line to line. All constants which appear in the formulation and the proof of the lemma will not change their value. This is because in the induction proof we need to know that constants do not depend on the parameter l.

Lemma 5.9. *Fix $t > 0$. Let $\mathcal{C}_3(t) = 16\tilde{c}^2 \mathcal{C}_1(t)$, where $\tilde{c}$ is a constant from Definition 1.4 and $\mathcal{C}_1(t)$ is a constant from Lemma 5.7. Let $n_0 \in \mathbb{N}$ (n_0 depends on t) be large enough so that*

$$L(n_0) \ge 1 \quad \text{and} \quad \frac{\mathcal{C}_3(t)}{L(n_0)} 8\mathcal{C}_2^{\frac{d-1}{2}} \sum_{p=1}^{\infty} \frac{1}{p^{1+\frac{\alpha}{2}}} \le 1, \tag{5.6}$$

where $\mathcal{C}_2$ is a constant from Lemma 5.8.

Then for $x \in R_n$, $k \ge n_0$, $n \ge k+2$, $l \ge 1$ we have

$$\mathbf{E}^x(S(n,k,l,t); e^{-\frac{1}{2}\int_0^{\sigma_k} V(X_s)ds}) \le \frac{\mathcal{C}_3(t)}{2^l} e^{-m^{1/\alpha}(n-k)} \frac{(\min\{n-k,k\})^{\frac{d-1}{2}}}{(n-k)^{\frac{d+\alpha+1}{2}} L(n)}. \tag{5.7}$$

Recall that (Definition 1.4) the function $L(n)$ is nondecreasing and tending to ∞, so for large enough n_0 (5.6) holds. We also point out that in the proof of Theorem 1.6 there will be some additional conditions on n_0.

Proof. First let $l = 1$. This is the case when the process makes only one appropriate "jump". Recall that $S(n,k,1,t) = \{X(\tau_n) \in R_k, \sigma_k < t\}$ and for $X(\tau_n) \in R_k$ we have $\tau_n = \sigma_k$. Note that for $s \in [0, \tau_n)$ we have $X_s \in A_n = (\overline{B(0, n-2)})^c$, so by Definition 1.5 $V(X_s) \ge L(n-2)$. Using Lemma 5.7 we obtain

$$\begin{aligned}\mathbf{E}^x(S(n,k,1,t);&e^{-\frac{1}{2}\int_0^{\sigma_k} V(X_s)ds}) \le \mathbf{E}^x(X(\tau_n) \in R_k, \tau_n < t; e^{-\tau_n L(n-2)/2}) \\ &\le \mathcal{C}_1(t) e^{-m^{1/\alpha}(n-k)} \frac{(\min\{n-k,k\})^{\frac{d-1}{2}}}{(n-k)^{\frac{d+\alpha+1}{2}}} \frac{2}{L(n-2)}.\end{aligned}$$

Recall that $n \ge k+2 \ge n_0+2$, so $L(n-1)$, $L(n-2)$ are no smaller than $L(n_0) \ge 1$. By Definition 1.4(3) $L(n) \le \tilde{c}L(n-1)+\tilde{c} \le 2\tilde{c}L(n-1)$. Similarly $L(n) \le 4\tilde{c}^2 L(n-2)$.

Therefore for $\mathcal{C}_3(t) = 16\tilde{c}^2\mathcal{C}_1(t)$ (5.7) holds for $l = 1$. We will show that for such $\mathcal{C}_3(t)$ (5.7) also holds for all $l \geq 1$. We will prove (5.7) by induction on l. Recall once again that in this proof we will not use our convention that the constants may change their value from one use to the next.

Let $l \geq 2$. Suppose we have proved (5.7) for $1, \dots, l-1$ and all $k \geq n_0$, $n \geq k+2$. We will show (5.7) for l. By Definition of $S(n, k, l, t)$ we have

$$\begin{aligned}
&\mathbf{E}^x(S(n,k,l,t); e^{-\frac{1}{2}\int_0^{\sigma_k} V(X_s)ds}) \\
&= \sum_{p=k+2}^{n-2} \mathbf{E}^x(\underbrace{S(n,p,l-1,t)}_{=A}, S(p,k,1,t); e^{-\frac{1}{2}\int_0^{\sigma_p} V(X_s)ds} e^{-\frac{1}{2}\int_{\sigma_p}^{\sigma_k} V(X_s)ds}) \\
&\leq \sum_{p=k+2}^{n-2} \mathbf{E}^x(A, S(p,k,1,t+\sigma_p); e^{-\frac{1}{2}\int_0^{\sigma_p} V(X_s)ds} e^{-\frac{1}{2}\int_{\sigma_p}^{\sigma_k} V(X_s)ds}).
\end{aligned}$$

By the strong Markov property the last expression is equal to

$$\sum_{p=k+2}^{n-2} \mathbf{E}^x(A; e^{-\frac{1}{2}\int_0^{\sigma_p} V(X_s)ds} \mathbf{E}^{X(\sigma_p)}(S(p,k,1,t); e^{-\frac{1}{2}\int_0^{\sigma_k} V(X_s)ds})).$$

Now by our induction hypothesis this is bounded from above by

$$\begin{aligned}
&\sum_{p=k+2}^{n-2} \frac{\mathcal{C}_3(t)}{2^{l-1}} e^{-m^{1/\alpha}(n-p)} \frac{(\min\{n-p,p\})^{\frac{d-1}{2}}}{(n-p)^{\frac{d+\alpha+1}{2}}} \frac{1}{L(n)} \\
&\qquad \times \frac{\mathcal{C}_3(t)}{2} e^{-m^{1/\alpha}(p-k)} \frac{(\min\{p-k,k\})^{\frac{d-1}{2}}}{(p-k)^{\frac{d+\alpha+1}{2}}} \frac{1}{L(p)}.
\end{aligned}$$

The function $L(n)$ is nondecreasing, so this is bounded from above by

$$\frac{\mathcal{C}_3(t)}{2^l} \frac{e^{-m^{1/\alpha}(n-k)}}{L(n)} \frac{\mathcal{C}_3(t)}{L(n_0)} \sum_{p=k+2}^{n-2} \frac{(\min\{n-p,p\})^{\frac{d-1}{2}}}{(n-p)^{\frac{d+\alpha+1}{2}}} \frac{(\min\{p-k,k\})^{\frac{d-1}{2}}}{(p-k)^{\frac{d+\alpha+1}{2}}}. \tag{5.8}$$

By Lemma 5.8 and (5.5) the last sum is smaller than

$$\begin{aligned}
&\sum_{p=k+2}^{n-2} \mathcal{C}_2^{\frac{d-1}{2}} \frac{(\min\{n-k,k\})^{\frac{d-1}{2}}}{(n-k)^{\frac{d+\alpha+1}{2}}} \frac{2^{1+\alpha/2}}{(\min\{n-p,p-k\})^{1+\frac{\alpha}{2}}} \\
&\leq 4\mathcal{C}_2^{\frac{d-1}{2}} \frac{(\min\{n-k,k\})^{\frac{d-1}{2}}}{(n-k)^{\frac{d+\alpha+1}{2}}} \sum_{p=k+2}^{n-2} \left(\frac{1}{(n-p)^{1+\frac{\alpha}{2}}} + \frac{1}{(p-k)^{1+\frac{\alpha}{2}}} \right) \\
&\leq 8\mathcal{C}_2^{\frac{d-1}{2}} \frac{(\min\{n-k,k\})^{\frac{d-1}{2}}}{(n-k)^{\frac{d+\alpha+1}{2}}} \sum_{p=1}^{\infty} \frac{1}{p^{1+\frac{\alpha}{2}}}.
\end{aligned}$$

Eventually, by (5.8) and (5.6) we obtain

$$\mathbf{E}^x(S(n,k,l,t); e^{-\frac{1}{2}\int_0^{\sigma_k} V(X_s)ds}) \leq \frac{\mathcal{C}_3(t)}{2^l} e^{-m^{1/\alpha}(n-k)} \frac{(\min\{n-k,k\})^{\frac{d-1}{2}}}{(n-k)^{\frac{d+\alpha+1}{2}} L(n)}.$$

$\square$

Now we can prove the main result of this paper.

Proof of Theorem 1.6. To prove intrinsic ultracontractivity of T_t we will use Condition 1.2, that is, for all $t > 0$ there exists c_t such that for all $x \in \mathbf{R}^d$, $T_t(1_{\mathbf{R}^d})(x) \le c_t T_t(1_D)(x)$, where $D = B(0,1)$.

First we estimate $T_t(1_{\mathbf{R}^d})(x)$. Fix $t > 0$. We assume that n_0 satisfies condtion (5.6) from Lemma 5.9. Let $n \ge 2n_0 + 4$ and $x \in R_n$. Applying the last lemma and (5.4) to the equality (5.3), we have

$$\begin{aligned} T_t(1_{\mathbf{R}^d})(x) \le\ & \mathbf{E}^x(\tau_{[n/2]} > t; e^{-\int_0^t V(X_s)ds}) \\ & + \sum_{k=n_0+2}^{[n/2]} \sum_{l=1}^{\infty} \frac{\mathcal{C}_3(t)}{L(n)} \frac{1}{2^l} e^{-m^{1/\alpha}(n-k)} e^{-\frac{t}{2}L(k-2)} \frac{k^{\frac{d-1}{2}}}{(n-k)^{\frac{d+\alpha+1}{2}}} \\ & + \sum_{k=n_0}^{n_0+1} \sum_{l=1}^{\infty} \frac{\mathcal{C}_3(t)}{L(n)} \frac{1}{2^l} e^{-m^{1/\alpha}(n-k)} \frac{k^{\frac{d-1}{2}}}{(n-k)^{\frac{d+\alpha+1}{2}}} = I + II + III. \end{aligned}$$

We have

$$I \le e^{-tL([n/2]-2)}.$$

Note that the function L is comparable on unit intervals (see Definition 1.4). Therefore there exists a constant $A > 0$ such that $L(n) \le e^{A\,n}$. We also have $L(n)/n \to \infty$. Let us recall that $t > 0$ is fixed. Thus we can choose n_0 (depending on t) large enough so that for $n \ge n_0$ we have $tL([n/2]-2) > m^{1/\alpha}n + (A+1)n$. For such n_0 we have

$$I \le e^{-tL([n/2]-2)} \le e^{-m^{1/\alpha}n} e^{-(A+1)n} \le \frac{c}{L(n)} e^{-m^{1/\alpha}n} \frac{1}{n^{\frac{d+\alpha+1}{2}}}.$$

By similar arguments we can choose n_0 large enough so that for $k \ge n_0$ we have $tL(k-2)/2 - m^{1/\alpha}k > k$. Note also that $k \le [n/2]$, so we have $n - k \ge n/2$. For such n_0 we have

$$II \le \frac{c_t}{L(n)} \frac{e^{-m^{1/\alpha}n}}{n^{\frac{d+\alpha+1}{2}}} \sum_{k=n_0+2}^{[n/2]} e^{-k} k^{\frac{d-1}{2}} \le \frac{c_t}{L(n)} e^{-m^{1/\alpha}n} \frac{1}{n^{\frac{d+\alpha+1}{2}}}.$$

For III we get similar estimates. It follows that for n_0 chosen as above, $x \in R_n$ and $n \ge 2n_0 + 4$ we have

$$T_t(1_{\mathbf{R}^d})(x) \le I + II + III \le \frac{c_t}{L(n)} e^{-m^{1/\alpha}n} \frac{1}{n^{\frac{d+\alpha+1}{2}}}.$$

Recall that by (5.6) we have $L(n_0) \ge 1$. By Definition 1.5 for $|x| \ge n_0$ we get that $V(x) \le CL(|x|) + C \le 2CL(|x|)$, so $L(|x|)$ and $V(x)$ are comparable. Since $n \ge |x| > n - 1$ ($x \in R_n$) we get

$$T_t(1_{\mathbf{R}^d})(x) \le \frac{c_t}{V(x)+1} e^{-m^{1/\alpha}|x|} \frac{1}{(|x|+1)^{\frac{d+\alpha+1}{2}}}, \tag{5.9}$$

for $|x| \ge 2n_0 + 4$.

Now we estimate $T_t(1_D)(x)$. Fix $t > 0$. Recall that $D = B(0,1)$ and let $|x| \ge n_0 + 1$, $B = B(x,1)$ (recall that $n_0 \ge 2$ so $|x| \ge 3$). We have

$$\begin{aligned} & T_t(1_D)(x) = \mathbf{E}^x(X_t \in D; e^{-\int_0^t V(X_s)ds}) \\ & \ge \mathbf{E}^x(X(\tau_B) \in D/2, \tau_B < t, \forall_{s \in [\tau_B, t+\tau_B]}\, X_s \in D; e^{-\int_0^{\tau_B} V(X_s)ds} e^{-\int_{\tau_B}^{t+\tau_B} V(X_s)ds}) \\ & \ge c_t \mathbf{E}^x(X(\tau_B) \in D/2, \tau_B < t, \forall_{s \in [\tau_B, t+\tau_B]}\, X_s \in D; e^{-\int_0^{\tau_B} V(X_s)ds}). \end{aligned}$$

By the strong Markov property this is equal to

$$\begin{aligned} &c_t \mathbf{E}^x(X(\tau_B) \in D/2, \tau_B < t; e^{-\int_0^{\tau_B} V(X_s)ds} \, \mathbf{P}^{X(\tau_B)}(\tau_D > t)) \\ &\geq c_t \mathbf{E}^x(X(\tau_B) \in D/2, \tau_B < t; e^{-\int_0^{\tau_B} V(X_s)ds}). \end{aligned}$$

For $y \in B$ we have $|y| < |x| + 1$, so for $s \in [0, \tau_B)$ we get $|X_s| < |x| + 1$. Since $|x| \geq n_0 + 1$ we have $L(|x| + 1) \geq 1$. Hence for $s \in [0, \tau_B)$ we have $V(X_s) \leq CL(|x| + 1) + C \leq 2CL(|x| + 1) \leq c' L(|x|)$. It follows that

$$\begin{aligned} T_t(1_D)(x) &\geq c_t \sum_{i=1}^{\infty} \mathbf{E}^x \left(X(\tau_B) \in D/2, \frac{t}{i+1} \leq \tau_B < \frac{t}{i}; e^{-\int_0^{\tau_B} V(X_s)ds} \right) \\ &\geq c_t \sum_{i=1}^{\infty} e^{-tc'L(|x|)/i} \mathbf{P}^x \left(X(\tau_B) \in D/2, \frac{t}{i+1} < \tau_B < \frac{t}{i} \right). \end{aligned}$$

By Lemmas 5.1 and 5.2 we get

$$\begin{aligned} T_t(1_D)(x) &\geq c_t \sum_{i=1}^{\infty} e^{-tc'L(|x|)/i} \frac{t}{i(i+1)} \frac{1}{|x|^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}|x|} \\ &\geq \frac{c_t}{|x|^{\frac{d+\alpha+1}{2}}} e^{-m^{1/\alpha}|x|} \frac{1}{tc'L(|x|) + 1} \\ &\geq \frac{c_t}{(|x| + 1)^{\frac{d+\alpha+1}{2}} (V(x) + 1)} e^{-m^{1/\alpha}|x|}. \end{aligned} \tag{5.10}$$

This and (5.9) gives Condition 1.2 for $|x| \geq 2n_0 + 4$. For small x this condition is obvious, because both sides are bounded, continuous and bounded away from 0. Therefore the semigroup T_t is IU.

When T_t is IU we can also show asymptotic behavior of the first eigenfunction $\phi_1(x)$:

$$\phi_1(x) = e^{-\lambda_1} T_1(\phi_1)(x) \leq e^{-\lambda_1} (\sup_{x \in \mathbf{R}^d} \phi_1(x)) T_1(1_{\mathbf{R}^d})(x) = C T_1(1_{\mathbf{R}^d})(x),$$

because $\phi_1(x)$ is bounded.

On the other hand for $D = B(0, 1)$ we have

$$\begin{aligned} \phi_1(x) &= e^{-\lambda_1} T_1(\phi_1)(x) \geq e^{-\lambda_1} T_1(1_D \phi_1)(x) \\ &\geq e^{-\lambda_1} (\inf_{x \in D} \phi_1(x)) T_1(1_D)(x) = c T_1(1_D)(x), \end{aligned}$$

because $\phi_1(x)$ is strictly positive, and continuous. Now (5.9) and (5.10) give (1.5) for large x ($|x| \geq 2n_0 + 4$). For small x (1.5) is trivial because ϕ_1 is continuous and strictly positive.

Finally we will show that if $V \in \mathcal{V}$ and condition $\lim_{|x| \to \infty} V(x)/|x| = \infty$ is not satisfied, then the semigroup T_t is not intrinsically ultracontractive. Since $V \in \mathcal{V}$ we have $\lim_{|x| \to \infty} V(x) = \infty$. So there exists $r_0 > 1$ such that for $|x| > r_0$ we have $V(x) > 1$. For such x we get $L(|x|) \leq V(x) \leq 2CL(|x|)$. L is comparable on unit intervals, so V is comparable on unit balls. Therefore we may assume that there exists a constant $M < \infty$ and a sequence of balls $B_n = B(x_n, 1)$, $|x_n| \to \infty$, $|x_n| > r_0 + 1$, $|x_n| \geq 3$, $n \geq 1$, such that $V(x) > 1$, $V(x)/|x| < M$ for any $x \in \bigcup_{n=1}^{\infty} B_n$.

Now we will show that Condition 1.3 does not hold. This implies that the semigroup T_t is not IU.

Let $D = B(0,1)$ and $x \in \bigcup_{n=1}^\infty B_n$. We have

$$T_t(1_D)(x) = \mathbf{E}^x(X_t \in D; e^{-\int_0^t V(X_s)ds}) \le \mathbf{P}^x(X_t \in D) = \int_D p(t,x,y)dy.$$

Since $|x_n| \ge 3$ we have $|x| \ge 2$ and $|x-y| \ge 1$ for $y \in D$. By Lemma 2.2 we get

$$T_t(1_D)(x) \le c_1 t e^{mt} |D| e^{-c_2(|x|-1)}.$$

To estimate $T_t(1_{\mathbf{R}^d})(x)$ we will again use the fact that V is comparable on unit balls. That is, there exists a constant c_3 such that for $x \in \bigcup_{n=1}^\infty B_n$ we have $\sup_{y \in B(x,1)} V(y) \le c_3 V(x)$. It follows that

$$\begin{aligned} T_t(1_{B(x,1)})(x) &\ge \mathbf{E}^x(\tau_{B(x,1)} > t; e^{-\int_0^t V(X_s)ds}) \\ &\ge \mathbf{P}^x(\tau_{B(x,1)} > t)e^{-tc_3 V(x)} \ge \mathbf{P}^0(\tau_{B(0,1)} > t)e^{-tc_3 M|x|} \\ &= c_t e^{-tc_3 M|x|}. \end{aligned}$$

If we choose t small enough to have $c_2 > tc_3 M$, then Condition 1.3 will not be satisfied for large enough x. This implies that the semigroup T_t is not IU. □

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